$[1,2]$-Connected domination number of graphs

G. Mahadevan[1] , **K. Renuka** [2] , **C.Sivagnanam** [3]

[1] Department of Mathematics, The Gandhigram Rural Institute - Deemed to be University, Gandhigram, Dindigul-624302, India.

E-mail address: drgmaha2014@ gmail.com.

[2] Ful ltime Research Scholar, Department of Mathematics, The Gandhigram Rural Institute - Deemed to be University, Gandhigram, Dindigul-624302, India.

E-mail address: math.renuka@ gmail.com

[3] Department of General Requirments, Sur College of Applied Sciences, Ministry of Higher Education, Sultanate of Oman.

E-mail address: choshi71 @ gmail.com

Abstract

A set $S \subseteq V(G)$ in a graph G is said to be a $[1,2]$-connected dominating set if for every vertex $v \in V - S$, $1 \leq |N(v) \cap S| \leq 2$ and $< S >$ is connected. The minimum cardinality of a $[1,2]$-connected dominating set is called the $[1,2]$-connected domination number and is denoted by $\gamma_{[1,2]c}(G)$. In this paper, we initiate a study of this parameter.

Keywords: Connected domination, [1,2]-sets, [1,2]-domination, [1,2]-connected domination

1 Introduction

The graph $G = (V, E)$ we mean a finite, undirected, connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. The degree of a vertex u in G is the number of edges incident with u and is denoted by $d_G(u)$, simply $d(u)$. The minimum and maximum degree of a graph G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1] and Haynes et.al [2].

A set $S \subseteq V$ is a dominating set if every vertex in $V - S$ is adjacent to atleast one vertex in S. The minimum cardinality of a dominating set is called the domination number and is denoted by $\gamma(G)$. Sampathkumar and Walikar $[6]$ introduced the concept of connected domination in graphs. A dominating set S is a connected dominating set if it induces a connected subgraph in G. The minimum cardinality of a connected dominating set of G is called the connected domination number and is denoted as $\gamma_c(G)$. Paulraj Joseph. J and Arumugam. S $[7,4]$ proved that $\gamma(G) + \chi(G) \leq n + 1$ and $\gamma(G) + \kappa(G) \leq n$. Also they characterized the corresponding extremal graphs.

Mustapha Chellali et.al., 2 first studied the concept of $[1,2]$-sets. A subset $S \subseteq V$ is a $[j,k] - set$ if, for every vertex $v \in V - S, j \leq |N(v) \cap S| \leq k$ for any non-negative integer j and k. A vertex set $S \subseteq V$ is a $[1,2]$-set if, $1 \leq |N(v) \cap S| \leq 2$ for every vertex $v \in V - S$, that is, every vertex $v \in V - S$ is adjacent to either one or two vertices in S. The minimum cardinality of a $[1,2]$-set of G is denoted by $\gamma_{[1,2]}(G)$ and is called $[1,2]$-domination number of G. Xiaojing Yang and Baoyindureng Wu $[8]$ extended the study of this parameter.

Motivated by the above concepts, in this paper we introduce the concept of $[1,2]$-connected domination in graphs.

Notations:

This research work was supported by Departmental Special Assistance, University Grants Commission, New Delhi and UGC-BSR Research fellowship in Mathematical Sciences-2014-2015.

Let H be a regular graph.

1. $H(P_k)$ is a graph obtained from H by attaching an end vertex of P_k to a vertex of H.
2. $H(m_1, m_2, \dots, m_n)$ denotes the graph obtained from the graph H by attaching m_i pendant edges to the vertex $v_i \in V(H), 1 \leq i \leq n$. The graph $K_2(m_1, m_2)$ is called bistar and it is also denoted by $B(m_1, m_2)$.
3. $H(mP_k)$ is the graph obtained from H by attaching m times an end vertex of P_k to a vertex of H.

2 Main Result

Definition 2.1 A set $S \subseteq V$ in a graph G is said to be a $[1,2]$-connected dominating set $([1,2]cd - set)$ if for every vertex $v \in V - S$, $1 \leq |N(v) \cap S| \leq 2$ and $< S >$ is connected. The minimum cardinality of a $[1,2]$-connected dominating set is called the $[1,2]$-connected domination number and is denoted by $\gamma_{[1,2]c}(G)$. A $[1,2]cd -$set of cardinality $\gamma_{[1,2]c}$ is called a $\gamma_{[1,2]c} -$set.

Observation 2.2
The $[1,2]$-connected domination number for some standard graphs can be easily found.

1. For a path P_n, $\gamma_{[1,2]c}(P_n) = \begin{cases} 1 & if\ n < 3 \\ n-2 & otherwise \end{cases}$
2. For a cycle C_n, $\gamma_{[1,2]c}(C_n) = n - 2, n \geq 3$.
3. If G is a complete graph K_n or a star $K_{1,n-1}$ or wheel W_n then $\gamma_{[1,2]c}(G) = 1$.
4. For a complete bipartite graph $K_{r,s}$, $\gamma_{[1,2]c}(K_{r,s}) = 2, r, s \geq 2$.
5. If G is a bistar $B(r,s)$, then $\gamma_{[1,2]c}(G) = 2$.

Theorem 2.3 For a tree T of order $n \geq 3$, $\gamma_{[1,2]c}(T) = n - e$, where e is the number of pendant vertices.

Proof. Let A be the set of all pendant vertices of T. Then $V - A$ is a $[1,2]$-connected dominating set of T. Hence $\gamma_{[1,2]c}(T) \leq n - e$. Let S be any $\gamma_{[1,2]c} -$ set of T. Since S is connected S contains all the internal vertices and hence $|S| \geq n - e$. Thus the result follows.

Corollary: 2.4 For a tree T, $\gamma_{[1,2]c}(T) = n - 2$ if and only if T is a path.
Observation: 2.5 Let G be a connected graph of order $n \geq 3$. Then $\gamma_{[1,2]c}(G) \leq n - 2$.
Observation: 2.6

1. The complement of a $[1,2]cd - set$ need not be a $[1,2]cd -$set.
2. Every $[1,2]cd -$set is a dominating set but not conversely.
3. Every $[1,2]cd -$set is a connected dominating set but the converse need not be true.

Observation: 2.7 For a graph G, $\gamma(G) \leq \gamma_{[1,2]}(G) \leq \gamma_{[1,2]c}(G)$ and $\gamma_c(G) \leq \gamma_{[1,2]c}(G)$.
Theorem: 2.8 For a graph G, $\lceil \frac{n}{\Delta+1} \rceil \leq \gamma_{[1,2]c}(G) \leq 2m - n$.

Proof. Since $\gamma(G) \leq \gamma_{[1,2]c}(G)$, the lower bound follows directly. By observation 2.5, $\gamma_{[1,2]c}(G) \leq n-2 \leq 2(n-1)-n \leq 2m-n$.

Theorem: 2.9 Let G be a connected graph. Then $\gamma_{[1,2]c}(G) = 2m-n$ if and only if $G \cong P_n$

Proof. If G is a path, then $\gamma_{[1,2]c}(G) = n-2 = 2(n-1)-n = 2m-n$. Conversly, assume that $\gamma_{[1,2]c}(G) = 2m-n$. Then $2m-n \leq n-2$ which gives $m = n-1$ and hence G is a tree. Thus $n-e = 2m-n = 2(n-1)-n$ which gives $e = 2$. Hence G is a path.

Observation: 2.10 If G is K_n or W_n or $K_{1,n-1}$ or C_n or a tree, then $\gamma_c(G) = \gamma_{[1,2]c}(G)$.
Observation: 2.11 For a graph G, $\gamma_{[1,2]c}(G) = 1$ if and only if there exist a vertex u such that $d(u) = n-1$.

3 Relationship with connectivity and chromatic number

Theorem 3.1 For a connected graph G, $\gamma_{[1,2]c}(G) + \kappa(G) \leq 2n-3$ and the equality holds if and only if G is K_3.

Proof. Since $\gamma_{[1,2]c}(G) \leq n-2$ and $\kappa(G) \leq n-1$ the result follows. Let $\gamma_{[1,2]c}(G) + \kappa(G) = 2n-3$. Then $\gamma_{[1,2]c}(G) = n-2$ and $\kappa(G) = n-1$. Since $\kappa(G) = n-1$, G is complete. But for a complete graph $\gamma_{[1,2]c}(G) = 1$ and hence $n = 3$. Thus G is K_3. The converse is obvious.

Theorem 3.2 Let G be a connected graph. Then $\gamma_{[1,2]c}(G) + \kappa(G) = 2n-4$ if and only if G is isomorphic to K_4 or P_3 or C_4.

Proof. Let $\gamma_{[1,2]c}(G) + \kappa(G) = 2n-4$. Then there are two cases to consider.
(i)$\gamma_{[1,2]c}(G) = n-2$ and $\kappa(G) = n-2$ (ii)$\gamma_{[1,2]c}(G) = n-3$ and $\kappa(G) = n-1$
Case 1.$\gamma_{[1,2]c}(G) = n-2$ and $\kappa(G) = n-2$
Since $\kappa(G) = n-2$ we have $n-2 \leq \delta(G)$. If $\delta = n-1$, then G is a complete graph, which is a contradiction. Hence $\delta(G) = n-2$. Then G is $K_n - Q$ where Q is a matching in K_n. Thus $\gamma_{[1,2]c}(G) \leq 2$. If $\gamma_{[1,2]c}(G) = 2$, then $n = 4$ and hence G is isomorphic to C_4. If $\gamma_{[1,2]c}(G) = 1$, then $n = 3$ and hence G is P_3.
Case 2.$\gamma_{[1,2]c}(G) = n-3$ and $\kappa(G) = n-1$
Since $\kappa(G) = n-1$, G is a complete graph. But for complete graph $\gamma_{[1,2]c}(G) = 1$ and hence $n = 4$. Thus G is K_4. The converse is obvious.

Theorem 3.3 For a cycle C_n, $\gamma_{[1,2]c}(C_n) = \chi(C_n)$ if and only if $n = 4 or 5$.

Proof. Let $\gamma_{[1,2]c}(C_n) = \chi(C_n)$. Then $\gamma_{[1,2]c}(C_n) = 2$ or 3. If $\gamma_{[1,2]c}(C_n) = 2$, then $n = 4$. If $\gamma_{[1,2]c}(C_n) = 3$, then $n = 5$. The converse is obvious.

Theorem 3.4 For a connected graph G, $\gamma_{[1,2]c}(G) + \chi(G) \leq 2n-2$ and the equality holds if and only if G is

K_3.

Proof. The inequality follows directly from $\gamma_{[1,2]c}(G) \le n-2$ and $\chi(G) \le n$. Let $\gamma_{[1,2]c}(G) + \chi(G) = 2n-2$. Then $\gamma_{[1,2]c}(G) = n-2$ and $\chi(G) = n$. Since $\chi(G) = n$, G is complete. But for complete graph $\gamma_{[1,2]c}(G) = 1$ and hence $n = 3$. Thus G is K_3. The converse is obvious.

Theorem 3.5 Let G be a connected graph. Then $\gamma_{[1,2]c}(G) + \chi(G) = 2n-3$ if and only if G is K_4 or P_3.

Proof. Let $\gamma_{[1,2]c}(G) + \chi(G) = 2n-3$. Then (i) $\gamma_{[1,2]c}(G) = n-2$ and $\chi(G) = n-1$ or (ii) $\gamma_{[1,2]c}(G) = n-3$ and $\chi(G) = n$.

Case:1 $\gamma_{[1,2]c}(G) = n-2$ and $\chi(G) = n-1$

Since $\chi(G) = n-1$, G contains a clique K on $n-1$ vertices or does not contains a clique K on $n-1$ vertices. Let G contains a clique K of order $n-1$vertices and let $v \notin V(K)$. Let $u \in N(v)$. Then $\{u\}$ is a $[1,2]cd$ $-$set of G. Thus $n = 3$ and $K = K_2$ and hence G is P_3.

If G does not contains a clique K on $n-1$ vertices, then it is verified that no graph exists.

Case:2 $\gamma_{[1,2]c}(G) = n-3$ and $\chi(G) = n$

Since $\chi(G) = n$ and hence G is complete. But for a complete graph $\gamma_{[1,2]c}(G) = 1$ and hence $n = 4$. Thus G is K_4. The converse is obvious.

Theorem 3.6 Let G be a connected graph. Then $\gamma_{[1,2]c}(G) + \chi(G) = 2n-4$ if and only if $G \in \{P_4, C_4, C_5, K_5, K_3(P_2), K_4 - e\}$.

Proof. Let $\gamma_{[1,2]c}(G) + \chi(G) = 2n-4$. This is possible only if (i)$\gamma_{[1,2]c}(G) = n-2$ and $\chi(G) = n-2$ or (ii)$\gamma_{[1,2]c}(G) = n-3$ and $\chi(G) = n-1$ or (iii)$\gamma_{[1,2]c}(G) = n-4$ and $\chi(G) = n$.

Case: 1 $\gamma_{[1,2]c}(G) = n-2$ and $\chi(G) = n-2$

Since $\chi(G) = n-2$, either G contains a clique K on $n-2$ vertices or $G = C_5 + K_{n-5}$. Let $G = C_5 + K_{n-5}$. If $n \ge 6$ then $\gamma_{[1,2]c}(G) = 1$ and hence $n = 3$ which is a contradiction. Thus $n = 5$ and hence $G = C_5$.

Suppose G contains a clique K on $n-2$ vertices. Let $S = V - V(K) = \{v_1, v_2\}$. Then either $< S >= K_2$ or $< S >= \overline{K}_2$.

Subcase: 1 $< S >= K_2$

Since G is connected either v_1 or v_2 is adjacent to a vertex in K. Let v_1 be adjacent to $u_1 \in V(K)$. Then $\{v_1, u_1\}$ is a $[1,2]cd$ $-$set of G. Hence $\gamma_{[1,2]c}(G) \le 2$ so that $n \le 4$. If $n = 4$, then G is either P_4 or C_4. If $n = 3$, then there is no graph satisfying the statement of the theorem.

Subcase: 2 $< S >= \overline{K}_2$

Since G is connected, we have two cases to consider.

Subcase: 2.1 $N(v_1) \cap N(v_2) \neq \phi$

Let $u \in N(v_1) \cap N(v_2)$. Then $\{u\}$ is a $[1,2]cd$ $-$set of G and hence $\gamma_{[1,2]c}(G) = 1$ which gives $n = 3$. Thus G is isomorphic to P_3. But $\chi(P_3) = 2 \neq n-2$ which is a contradiction.

Subcase: 2.2 $N(v_1) \cap N(v_2) = \phi$

Let $v_1u_1, v_2u_2 \in E(G)$ for some $u_1, u_2 \in V(K)$. Then $\{u_1, u_2\}$ is a $[1,2]cd$ $-$set of G. Thus $\gamma_{[1,2]c}(G) = 2$ and hence $n = 4$. Thus $K = K_2$ which gives $G = P_4$.

Case:2 $\gamma_{[1,2]c}(G) = n-3$ and $\chi(G) = n-1$

Since $\chi(G) = n-1$, either G contains a clique K on $n-1$ vertices or does not contains a clique K on $n-1$ vertices. Let G contains a clique K on $n-1$ vertices and let $v \in V - V(K)$. Since G is connected, without loss of generality we may assume that v be adjacent to $u \in V(K)$. Then $\{u\}$ is a $[1,2]cd -$set of G which gives $\gamma_{[1,2]c}(G) = 1$ and hence $n = 4$. Thus $K = K_3$. If $d(v) = 1$ then $G \cong K_3(P_2)$. If $d(v) = 2$ then $G = K_4 - e$.

If G does not contains a clique K on $n-1$ vertices, then it is verified that no graph exist.

Case: 3 $\gamma_{[1,2]c}(G) = n-4$ and $\chi(G) = n$

Since $\chi(G) = n$, G is complete and hence $\gamma_{[1,2]c}(G) = 1$. Thus $n = 5$, so that $G = K_5$. The converse is obvious.

Theorem 3.7 Let G be a connected graph then, $\gamma_{[1,2]c}(G) + \chi(G) = 2n-5$ if and only if G is isomorphic to P_5 or $K_3(P_3)$ or $K_{1,3}$ or $K_3(1,1,0)$ or K_6 or $K_5 - Y$, where $Y \subseteq E(K_5)$ such that the edge induced subgraph $< Y >$ is a star and $1 \leq |Y| \leq 3$ or $H_i, 1 \leq i \leq 3$.

Proof. Let $\gamma_{[1,2]c}(G) + \chi(G) = 2n-5$. This is possible only if (i) $\gamma_{[1,2]c}(G) = n-2$ and $\chi(G) = n-3$ or (ii) $\gamma_{[1,2]c}(G) = n-3$ and $\chi(G) = n-2$ or (iii) $\gamma_{[1,2]c}(G) = n-4$ and $\chi(G) = n-1$ or (iv) $\gamma_{[1,2]c}(G) = n-5$ and $\chi(G) = n$.

Case.1 $\gamma_{[1,2]c}(G) = n-2 and \chi(G) = n-3$

Since $\chi(G) = n-3$, G contains a clique on $n-3$ vertices or does not contains a clique on $n-3$ vertices. Suppose G contains a clique K on $n-3$ vertices. Let $S = V(G) - V(K) = \{v_1, v_2, v_3\}$.

Subcase 1: $< S >= \overline{K}_3$

Since G is connected, every vertex in S is adjacent to atleast one vertex in K.

Suppose all the vertices of S are adjacent to $u_1 \in K$. Then $\{u_1\}$ is $[1,2]cd -$set of G. Hence $\gamma_{[1,2]c}(G) = 1$. Then $n = 3$ which is a contradiction.

Suppose v_1, v_2 are adjacent to a vertex $u_1 \in K$ and $v_3 u_2 \in E$. Then $\{u_1, u_2\}$ is $[1,2]cd -$set. Hence $\gamma_{[1,2]c}(G) \leq 2$, then $n \leq 4$ and hence $K = K_1$, which is a contradiction.

Suppose all the vertices of S are adjacent to distinct vertices of K. Let $u_i v_i \in E$. Then $\{u_1, u_2, u_3\}$ is $[1,2]cd -$set. Hence $\gamma_{[1,2]c}(G) \leq 3$, then $n \leq 5$, which is a contradiction.

Subcase 2: $< S >= K_2 \cup K_1$

Let $v_1 v_2 \in E$. Since G is connected v_1 and v_3 have neighbors in K.

Suppose $N(v_1) \cap N(v_3) \neq \emptyset$. Let $v_1 u_1, v_3 u_1 \in E$. Then $\{v_1, u_1\}$ is a $[1,2]cd -$set of G and hence $\gamma_{[1,2]cc}(G) \leq 2$ and $n \leq 4$, which is a contradiction.

Suppose $N(v_1) \cap N(v_3) = \emptyset$. Let $v_1 u_1, v_3 u_2 \in E$. Then $\{v_1, u_1, u_2\}$ is a $[1,2]cd -$set of G and hence $n = 5$.Then $G \cong P_5$. If $d(v_1) > 2$ or $d(v_2) > 1$ or $d(v_3) > 1$, then no graph exists.

Subcase 3: $< S >= C_3$

Let $v_1 u_1 \in E$. Then $\{u_1, v_1\}$ is a $[1,2]cd -$set of G and hence $\gamma_{[1,2]c}(G) \leq 2$. Then $n \leq 4$. Hence $K = K_1$, which is a contradiction.

Subcase 4: $< S >= P_3$

Let $< S >= (v_1, v_2, v_3)$. Since G is connected, at least one vertex of S is adjacent to a vertex in K. Let $v_2 u_1 \in E(G)$. Then $\{v_2, u_1\}$ is a $[1,2]cd -$set of G. Thus $\gamma_{[1,2]c}(G) \leq 2$ and hence $n = 4$. Then $K = K_1$ and hence no graph exists.

Suppose $v_2 u_1 \notin E(G)$. Let $v_1 u_1 \in E(G)$. Then $\{v_1, v_2, u_1\}$ is $[1,2]cd -$set of G. Hence $\gamma_{[1,2]c}(G) \leq 3$ and $n \leq 5$. It is clear that $n = 5$. Then $K = K_2$ and $G \cong P_5$. If $d(v_1) > 2$ or $d(v_3) > 1$, then no graph exists.

If G does not contains a clique on $n-3$ vertices, it can be verified that no graph exists.

Case.2 $\gamma_{[1,2]c}(G) = n-3 and \chi(G) = n-2$

Since $\chi(G) = n-2$, G contains a clique K of order $n-2$ or $G = C_5 + K_{n-5}$. Let K be a clique of order $n-2$ in G. Let $S = V - V(K) = \{v_1, v_2\}$

Subcase.1 $< S >= K_2$

Without loss of generality we assume that $v_1u_1 \in E$ for some $u_1 \in V(K)$. Then $\{v_1, u_1\}$ is a $[1,2]cd-$set of G and hence $\gamma_{[1,2]c}(G) \leq 2$. If $\gamma_{[1,2]c}(G) = 1$ then $n = 4$ and hence $K = K_2$ which is a contradiction. If $\gamma_{[1,2]c}(G) = 2$, then $n = 5$ and $K = K_3$. Hence G is isomorphic to $K_3(P_3)$ or $H_1 or H_2 or H_3$.

Subcase.2 $< S >= \overline{K}_2$

If $N(v_1) \cap N(v_2) \neq \phi$, then $\gamma_{[1,2]c}(G) = 1$. Hence $n = 4$ and $K = K_2$. Thus $G \cong K_{1,3}$. If $d(v_1) \geq 2$, then $K = K_3$ which is a contradiction. Thus $d(v_1) = 1$ and hence $G \cong K_{1,3}$.

Let $N(v_1) \cap N(v_2) = \phi$. Let $u \in N(v_1)$ and $v \in N(v_2)$. Then $\{u, v\}$ is a $\gamma_{[1,2]c}-$set of G. Thus $n = 5$ and $K = K_3$. Hence G is isomorphic to either $K_3(1,1,0)$ or H_1.

Suppose $G = C_5 + K_{n-5}$. Then $\gamma_{[1,2]c]}(G) = 1$ and hence $n = 4$ which is a contradiction.

Case.3 $\gamma_{[1,2]c}(G) = n-4 \text{ and } \chi(G) = n-1$

Then G contains a clique K of order $n-1$. Let $V - V(K) = \{v\}$. Since G is connected there exists a vertex $u \in V(K)$ such that $uv \in E$. Then $\{u\}$ is a $[1,2]cd-$set of G. Thus $n = 5$ and $K = K_4$. Hence G is isomorphic to $K_5 - Y$ where $Y \subseteq E(K_5)$ such that the edge induced subgraph $< Y >$ is a star and $1 \leq |Y| \leq 3$.

Case.4 $\gamma_{[1,2]c}(G) = n-5 \text{ and } \chi(G) = n$

Since $\chi(G) = n$, G is a complete graph. But for complete graph G, $\gamma_{[1,2]c}(G) = 1$, so that $n = 6$ and hence $G \cong K_6$. The converse is obvious.

Conclusion:

In this paper, we introduced the concept of $[1,2]$-Connected domination number of graphs and obtained its bounds. We also showed the relation between $[1,2]cd-$set with connectivity and chromatic number of graphs.

References

[1] G.Chartrand and L. Lesniak, Graphs and Digraphs, Fourth Edition CRC Press, Boca Raton, 2005.

[2] T.W.Haynes, S.T.Hedetniemi and P.J.slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, 1998.

[3] Mustapha Chellali, Teresa W. Haynes, Stephen T. Hedetniemi and Alice McRae, [1,2]-Sets in graphs, Discrete Applied Mathematics, 161, (2013), 2885 − 2893.

[4] J.Paulraj Joseph and S.Arumugam, Domination and Connectivity in graphs, International Journal of Management and Systems, 8, (1992), 233 − 236.

[5] J.Paulraj Joseph and S.Arumugam, Domination and coloring in graphs, International journal of Management systems, 8(1), (1997), 37 − 44.

[6] E.Sampathkumar and H.B.Walikar, The connected domination number of a graph, Journal of Math. Phy. Sci. Vol. 13, No. 6, (1979), 607 − 613.

[7] C.Sivagnanam, Double domination number and connectivity of graphs,International Journal of Digital Information and Wireless Communications, 2(1),(2012), 40 − 45.

[8] Xiaojing Yang and Baoyindureng Wu, [1,2]-domination in graphs, Discrete Applied Mathematics, 175, (2014), 79 − 86.