Exponential stability and synchronization results for Caputo fractional order impulsive differential equation

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Abstract: This paper studies exponential stability results for Caputo fractional order impulsive differential equations. We derive some new sufficient conditions for the given system using Convex Lyapunov function and matrix inequality approach. In addition we have incorporated the discussion of exponential synchronization for the error system derived from the given drive system and its response system. The obtained results are validated through fractional order Lorenz system

Keywords: Lyapunov function, matrix inequality approach, exponential stability, exponential synchronization, Caputo fractional order impulsive differential equation.

1. INTRODUCTION

In the recent years, the nonlinear fractional differential equations (NFDEs) remarkably received notable attention due to its application in electronics, bio engineering, epidemiology, physics, engineering, biology, etc. In fact, the nonlinear fractional differential equations exhibit chaos, for example, Lorenz system[22], Lu system[23], Chen system[24], etc. Therefore, the investigation of stability of nonlinear differential equations of fractional order is unavoidable.

On the contrary, controllers play an vital role in the chaotic fractional differential equations. Al- ready some valuable results have been obtained for stability and synchronization based in numerous controllers. In[9] using a convex and positive definite Lyapunov function with fractional order derivative as negative definite, it is revealed that the fractional order system is Mittag Leffler stable. [25] and [26] proposes certain stability conditions through matrix inequality for fractional order systems.[27] and [28] discusses the designing of feedback controllers for stabilization of fraction order systems using LMI conditions. In [17] the synchronization is investigated and based on Lyapunov stability a different fractional order controller for synchronization which is hyper chaotic is studied. In [18] the synchronization is investigated among fractional order hyper-chaotic systems and hyper chaotic integer order systems through a sliding mode type of controller and through proper drive, response system and parameters.

However, the impulse effects widely exist while investigating nonlinear differential equations whose order is fractional. Impulses can make sudden changes in the nature of the systems and many researchers contribute to initiate their research in this area. Recently, impulsive effects have been used as control point of view, for example neural networks[11], epidemic models[12], biological models[14], financial models [13], economic models etc. It is a recently developed branch of control theory and is important in secure communication[8]. In [29] uses impulsive controller to stabilize chaotic behavior of fractional order system. In [20] re-evaluates some of the important conclusions made on the stability of impulsive systems. In [30] by applying Lyapunov stability and LMI, the impulsive synchronization of different and same structure chaotic fractional order system is studied.

Inspired from the above mentioned results this paper focus a computationally strong approach to construct an impulsive stabilizing and an impulsive synchronizing controller for nonlinear fractional impulsive differential equation. The central features of this work are summarized as follows

(i) By using Lyapunov convex function and matrix inequality approach, we derive some new sufficient conditions for exponential stability and exponential synchronization for nonlinear fractional order impulsive differential equations.

(ii) The derived results are new and better than past literature[17][27][28].

The rest of this paper is as follows through some notations, definitions and lemmas in the preliminary section 2. In section 3, we prove exponential stability and exponential synchronization of fractional order impulsive differential equation. In section 4, proposed results are validated by an illustrative example that is Lorenz system. We end the paper by conclusion

Notations

$\mathbb{R}^+=\{x \in \mathbb{R} : x \geq 0\}$ where R is the the set of real numbers, $^+ = \{1, 2, 3, ...\}$, $\mathbb{R}^n$ is the Euclidean space. $\mathbb{R}^{n\times n}$is the set of all real matrices, $\|.\|_2$ is the 2-norm where $\|.\|_2 : \mathbb{R}^n \to \mathbb{R}$

2. PRELIMINARIES

Definition1. The Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$on the half axis $\mathbb{R}^+$ is defined as

$$ {}_{t_0}^{c}D_t^{\alpha}\left(g(t)\right) = \frac{1}{\Gamma(n-a)} \int_{t_0}^{t} \frac{g^{(n)}(\tau)}{(t-\tau)^{(\alpha-n+1)}} d\tau $$

with $n = \min\{ k \in N | k > \alpha > 0 \}$, where $g^{(n)}(t)$ is the n-order derivative of $g(t)$ and $\Gamma(.)$ is the Gamma function[9].

Definition2. The Riemann Liouville Fractional derivative of order α of function $g(t)$ is defined as

$$ {}_{t_0}^{R}D_t^{\alpha}\left(g(t)\right) = \frac{1}{\Gamma(n-a)} \frac{d^n}{dt^n} \int_{t_0}^{t} t - \tau^{(\alpha-n-1)}\, g(\tau)\, d\tau $$

Where,$n-1 \leq \alpha < n$; $n \in \mathbb{Z}^+$.

Model Description

Consider the fractional order impulsive control Caputo fractional differential equation of the following form

$$ \begin{aligned} &{}_{t0}^{C}D_t^{\alpha}\, x(t)= Ax(t) + Bf(x(t)); \; t\neq t_k ,\; t\geq t_0 \\ &x(t_k^-)= B_k (x(t_k^-)); \; t = t_k,\; k \in \mathbb{Z}^+ \qquad (1) \\ &x(t_0) = x_0, \end{aligned} $$

where $\alpha \in (0,1)$ and A,B $\in \mathbb{R}^{nxn}$ are constant matrices, $f(x(t))$:$\mathbb{R}^n \to \mathbb{R}$ is the non-linear function vector, $B_k \epsilon\, \mathbb{R}^{nxn}$ is impulsive gain matrix, $t_1 < t_2 < t_3 < \ldots\ldots.. < t_k$ with $\lim_{k\to\infty} tk = +\infty$. We assume that x(t) is right continuous at $t = t_k$ and $x(t_k^+) = x(t_k^-)$.

Assumption 1: f(x(t)) is continuous and satisfied Lipschitz condition on $\mathbb{R}^n$.

Lemma 1:[16] For the given vectors x,y ϵ $\mathbb{R}^n$ and any positive constant $\mathcal{E} > 0$, the following inequality holds:

$$2x^T(t)y(t) \leq \varepsilon x^T(t)x(t) + \varepsilon^{-1}y^T(t)y(t)$$

Lemma 2:[9]Let $\omega \in \mathbb{R}^n$. If $V(h(t)) : \omega \rightarrow \mathbb{R}$ and $h(t) : [0,\infty) \rightarrow \omega$ are two continuous and differentiable functions and V(h(t)) is convex over ω, then

$$_{t0}^{C}D_t^{\alpha}\ V(h(t)) \leq \frac{\partial V^T}{\partial h}\ _{t0}^{C}D_t^{\alpha}\ (h(t)),\ \forall\ \alpha \in (0,1),\ \forall\ t \geq 0$$

.Specially for any P > 0, when

$$h(t) = h^T(t)Ph(t)$$

then the following well known inequality holds;

$$_{t0}^{C}D_t^{\alpha}\ (h^T(t)Ph(t)) \leq 2h^T(t)P\, _{t0}^{C}D_t^{\alpha}\ (h(t)).$$

Lemma 3: For all $a \in \mathbb{R}$ and a real valued continuous function G(t) on $[a,\infty)$, if there exist a constant θ such that $_{t0}^{C}D_t^{\alpha}\left(G(t)\right) < \theta\, G(t);\ 0 < \alpha \leq 1$ then,

$$G(t) \leq G(a)e^{\int_a^t \frac{\theta}{\Gamma\alpha}(t-\tau)^{\alpha-1}}\, d\tau$$

$$= G(a)e^{\frac{\theta}{\Gamma\alpha-1}(t-a)^{\alpha}}$$

3.MAIN RESULT

Using convex Lyapunov function, exponential stability of nonlinear Caputo fractional order system is studied.

Exponential stability results

Theorem 3.1: Let Q be a positive definite matrix. If there exists $\epsilon, \gamma, \mu > 0$ and $\zeta^k > 1$ such that the following conditions

I. $QA + A^TQ + \epsilon QBB^TQ + \epsilon^{-1}L^2 \leq -\gamma Q$,

II. $B_k^T Q B_k < e^{-\mu}Q$,

III. $\zeta^k\ e^{-\mu_k}\ e^{-\frac{\eta}{\Gamma\alpha+1}(t_k - t_{k-1})^{\alpha}} < 1$,

are satisfied. Then the system (1) is exponentially stable.

Proof. By assumption 1, the solution of equation (1) is piece-wise right continuous function. Consider the convex Lyapunov function

$$W(t) = y^T(t)Qy(t)$$

Taking Caputo derivative and by using Lemma 2, when $t \in (t_k, t_{k+1})$ for $k \in Z^+$

$$ {}_{t0}^{C}D_t^{\propto}(W(t)) \leq 2y^T(t)Q\{{}_{t0}^{C}D_t^{\propto}(y(t)) = 2y^T(t)Q[Ay(t)+Bf(y(t))] $$
$$ = 2y^T(t)QAy(t)+ 2y^T(t)QBf(y(t)) $$
$$ \leq y^T(t)QAy^T(t)+ y(t)A^TQy(t) + 2y^T(t)QBf(y(t)) $$

$$ {}_{t0}^{C}D_t^{\propto}(W(t)) \leq y^T(t)QAy(t) + y^T(t)A^T Qy(t) + y^T(t)QBB^T Qy(t) + e^{-1}f^T(y(t))f(y(t)) $$

By assumption 1, it follows that

$$ {}_{t0}^{C}D_t^{\propto}(W(t)) = y^T(t)QAy(t) + y^T(t)A^T Qy(t) + y^T(t)QBB^T Qy(t) + e^{-1} Ly^T(t)Ly(t) $$
$$ = y^T(t)QAy(t) + y^T(t)A^T Qy(t) + y^T(t)QBB^TQy(t) + e^{-1}y^T(t)Ly(t) $$
$$ = y^T(t)[QA+A^TQ+QBB^TQ+e^{-1}L^2]y(t) $$

Now take $QA + A^TQ + QBB^TQ + e^{-1} L^2 \leq -\gamma Q$

Thus

$$ {}_{t0}^{C}D_t^{\propto}(W(t)) \leq y^T(-\gamma Q)y(t) $$
$$ \leq -\gamma y^T Qy(t) $$

That is

$$ {}_{t0}^{C}D_t^{\propto}(W(t)) \leq -\gamma W(y(t)) $$

By using lemma 3, we have

$$ W(t) \leq W(t_k)\, e^{\frac{-\eta}{\Gamma(\propto+1)}(t-t_k)^{\propto}} \quad , t \in (t_k,t_{k+1}) $$

When $t = t_k$ it follows from second equation of system (1) that

$$ W(t_k) = y^T(t_k)Qy(t_k) $$
$$ = [(B_k)y(t_k^-)]^T Q(B_k)y(t_k^-) $$
$$ = y^T(t_k^-)(B_k)^T Q(B_k)y(t_k^-) $$

Now by taking $(B_k)^TQ(B_k) < e^{-\mu}Q$

Then
$$ \boldsymbol{W(t_k) \leq y^T(t_k^-)e^{-\mu}Qy(t_k^-)} $$
$$ \boldsymbol{= e^{-\mu}y^T(t_k^-)Qy(t_k^-)} $$

Thus

$$W(t_k) \leq e^{-\mu} W(t_k^-)$$

Now by using lemma 3 we can write for any $t \epsilon (t_0, t_1)$, *we have*

$$W(t) \leq W(t_0) e^{\frac{-\gamma}{\Gamma(\alpha+1)}(t-t_0)^\alpha},$$

which leads to:

$$W(t_1^-) \leq W(t_0) e^{\frac{-\gamma}{\Gamma(\alpha+1)}(t_1-t_0)^\alpha}$$

Next for any $t \epsilon (t_1, t_2)$,

$$W(t) \leq W(t_1) e^{\frac{-\gamma}{\Gamma(\alpha+1)}(t-t_1)^\alpha}$$

$$\leq W(t_0) e^{-\mu} e^{\frac{-\gamma}{\Gamma(\alpha+1)}[(t-t_1)^\alpha + (t_1-t_0)^\alpha]}$$

Similarly, for any $t \epsilon (t_k, t_{k+1})$ *we have,*

$$W(t) \leq W(t_k) e^{\frac{-\gamma}{\Gamma(\alpha+1)}(t-t_k)^\alpha}$$

$$\leq \; W(t_0) \prod_{i=1}^{k} e^{-\mu i} e^{\frac{-\gamma}{\Gamma(\alpha+1)}(t_i-t_{i-1})^\alpha} \times e^{\frac{-\gamma}{\Gamma(\alpha+1)}(t-t_k)^\alpha}$$

Now by taking $\zeta^k \; e^{-\mu_k} \; e^{-\frac{\eta}{\lceil\alpha+1}(t_k-t_{k-1})^\alpha} < 1$,

we get

$$W(t) \leq W(t_0) \frac{1}{\zeta^k} e^{\frac{-\gamma}{\Gamma(\alpha+1)}(t-t_k)^\alpha}$$

Where $\frac{1}{\zeta^k} \to 0$ *as* $k \to \infty$, *then* $W(t) \leq 0$.

Then system 1 is exponentially stable.

3.2 Exponential Synchronization

Consider drive system

$${}_{t_0}^{C}D_t^{\alpha}\, x(t) = \text{A x(t) + B f(x(t))} \; ; \; t = t_k \, , \; t \geq t_0$$

$$x(t_k^{\,+}) = C\, x(t_k^{\,-}) \; ; \; t = t_k$$

$$x(t_0) = x_0 \qquad (4)$$

The corresponding response system can be described as follows

$${}_{t_0}^{C}D_t^{\alpha}\, y(t) = \text{A y(t) + B f(y(t))} ;\ t = t_k ,\ t \geq t_0$$
$$y(t_k^{+}) = C\, y(t_k^{-}) ;\ t = t_k$$
$$y(t_0) = y_0 \qquad (5)$$

Define the error variable as θ(t) = y(t) – x(t). Then we obtained error system from (5) – (4), it defined as

$${}_{t_0}^{C}D_t^{\alpha}\, \theta(t) = A\, \theta(t) + B\, f(\theta(t)) ;\ t = t_k ,\ t \geq t_0$$
$$\theta(t_k^{+}) = C\, \theta(t_k^{-}) ;\ t = t_k$$
$$\theta(t_0) = \theta_0 \qquad (6)$$

where f(θ(t)) = f($\theta(t)$ + x(t)) – f(x(t)).

Theorem 3.2: Let Q be a positive definite matrix. If there exists $\gamma, \mu_k > 0$ and $\zeta^k > 1$ such that the following conditions

I. $QA + A^T Q + \epsilon QBB^T Q + \epsilon^{-1} L^2 \leq -\gamma Q$,

II. $B_k^T Q B_k < e^{-\mu} Q$,

III. $\zeta^k\, e^{-\mu_k}\, e^{-\frac{\eta}{\lceil \alpha+1}(t_k - t_{k-1})^{\alpha}} < 1$,

are satisfied. Then the system (6) is exponentially synchronized.

PROOF: The proof is similar to Theorem 3.1, thus we omit it.

4. NUMERICAL EXAMPLES

In this section, we give two illustrative examples to verify the effectiveness of exponential stability and exponential synchronization results obtained in the previous section.

Example 1: Consider the three- dimensional Lorenz system

$${}_{t_0}^{C}D_t^{\alpha}\, x_1(t) = 10(x_2 - x_1) ,$$
$${}_{t_0}^{C}D_t^{\alpha}\, x_2(t) = -28x_1 - x_2 - x_1 x_3$$
$${}_{t_0}^{C}D_t^{\alpha}\, x_3(t) = x_1 x_2 - \frac{8}{3} x_3$$

The above system can be represented as a fractional order impulsive control as follows

$${}_{t_0}^{C}D_t^{\alpha}\, x(t) = \text{A x(t) + B f(x(t))} ;\ t = t_k ,\ t \geq t_0$$
$$x(t_k^{+}) = C\, x(t_k^{-}) ;\ t = t_k$$

$$x(t_0) = x_0 \epsilon \mathbb{R}^3 \quad (7)$$

where $0 < \alpha < 1, x(t) = (x_1(t), x_2(t), x_3(t)) \epsilon \mathbb{R}^3$, and

$$A = \begin{bmatrix} -10 & 10 & 0 \\ -28 & -1 & 0 \\ 0 & 0 & \frac{8}{3} \end{bmatrix}, \quad f(x(t)) = \begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

Note that, the initial conditions are chosen as $x_0 = [0.1 \quad 0.1 \quad 0.1]^T$. By using Theorem 3.1, we found that the matrix inequality conditions are satisfied with Lipschitz constant L= 0.5, ϵ = 0.01, μ_k = 0.5, γ = 0.01 and we get

$$P = \begin{bmatrix} 0.0697 & 0.0221 & 0 \\ 0.0221 & 0.1855 & 0 \\ 0 & 0 & 0.1263 \end{bmatrix}$$

Therefore, the systems (7) achieves exponential stability and similarly achieve exponential synchronization by Theorem 3.2.

5. Conclusion

In this paper, we have obtained exponential stability and exponential synchronization results for nonlinear fractional order impulsive differential equations. By using the convex Lyapunov function and matrix inequality approach we achieved the desired conditions for given system

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