Research Article

Applications of Triple Laplace Transformation to Volterra Integro Partial Differential Equation

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Abstract: Present study is associated with verifying convergence property of Triple Laplace transform (TLT). In this work the new theorem is proposed to verify the convergence. The TLT is applied on the function and the result is verified with standard result and TLT is also applied to verify solution of Volterra-integro partial differential Equation (VIPDE) under certain initial conditions and the result obtained are found comparable with standard.

Key words: Triple Laplace Transform (TLT), Convergence, Volterra Integral Partial Differential Equation (VIPDE).

1. Introduction

Partial differential equations (PDEs) play a very important role in the real life problems (**Widder, 2005)**, but PDEs are much harder to solve than ordinary differential equations. There are many PDEs like Wave equations, Heat equation, Laplace equation and Integro differential equations etc **(Gupta et al. , 2013),[Rogers], [Wazwaz, 2010]**. Integro differential equations have many applications in Engineering, Physics, Chemistry and Mathematics. Particularly Volterra – integro is one of the important differential equations which play the role in nuclear reactions , circuit analysis, glass forming process, nano hydrodynamics etc.

There are many methods used for solution of VIPDEs. For example Volterra – integro differential Equation has been solved by He's Homotopy Perturbation Method **(Shhed, 2005)** and **Moghadam** used the differential transforms, **Fahim et al.** used sinc-collocation method and **Abdul-Majid Wazwaz** used combined Laplace transform–Adomian decomposition method.

The Laplace transformation is a very useful and effective technique for solving such type of partial differential equations with initial and boundary value problems and mainly utilized in engineering purposes for system modeling in which a large differential equation must be solved, which was introduced by Laplace in 1782 by **Widder.**

In 2008 **Adam Kilicman** extended the Laplace transform to the concept of double Laplace transform . This concept has been successfully used for solving some kind of differential equations **(Eltayeb et al. 2013)(M. Idrees, 2018),(Ozel, 2012)**. Recently in 2013, **Abdon Atangana** also extended the double Laplace transform to the concept of triple Laplace transform and this new concept of triple Laplace transform, also works very effectively for solving such kind of partial differential equation involving triple integrals by **(Elzaki, 2019), (Khan et al. , 2020),(Khan et al. , 2018), (Mousa et al. , 2019).**

The main aim of this research work is to extend the concepts of triple Laplace transform and to solve Volterra-integro partial differential equations using triple Laplace transformation.

2. Basic Definition and Theorems

Definition 2.1 ***Laplace Transform*** *: The Laplace transform denoted by the operator* $\mathrm{L}(.)$ *defined by the integral Equation.*

$$L[f(t)] = F(S) = \int_0^\infty f(t)e^{-st}dt, 0 \le t < \infty$$

Definition 2.2 ***Triple Laplace Transform****: Let f(x,y,t) be a continuous function that can be expressed as convergent infinite series , then triple Laplace transform of f(x,y,t) is defined as*

$$L[f(x,y,t)] = F(\sigma,\rho,\delta) = \int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma x-\rho y-\delta t} f(x,y,t)dxdydt \quad (1)$$

Where $x, y, t > 0$ and $\sigma,\ \rho,\ \delta$ are Laplace variables and

$$f(x,y,t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\sigma x}\left[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}e^{\rho y}\left(\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}e^{\delta t}f(\sigma,\rho,t)d\delta\right)d\rho\right]d\sigma$$

is the inverse Laplace transform denoted by $L^{-1}_{x,y,t}$.

Theorem 2.3 *The integral*

$$\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma x-\rho y-\delta t}H(x,y,t)dxdydt \quad (2)$$

Converges at $\sigma<\sigma_0,\ \rho<\rho_0$ and $\delta<\delta_0$.

If $H(x,y,t)$ is continuous function in the positive of the x,y,t plane and it's integral converges at $\sigma=\sigma_0,\ \rho=\rho_0$ and $\delta=\delta_0$.

Proof. We have

$$\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma x-\rho y-\delta t}H(x,y,t)dxdydt$$

$$=\int_0^\infty e^{-\delta t}\{\int_0^\infty e^{-\sigma x}\left[\int_0^\infty e^{-\rho y}H(x,y,t)dy\right]dx\}dt$$

$$=\int_0^\infty e^{-\delta t}\left[\int_0^\infty e^{-\sigma x}G(x,\rho,t)dx\right]dt$$

$$=\int_0^\infty e^{-\delta t}K(\sigma,\rho,t)dt$$

Where

$$G(x,\rho.t)=\int_0^\infty e^{-\rho y}H(x,y,t)dy$$

and

$$K(\sigma,\rho,t)=\int_0^\infty e^{-\sigma x}G(x,\rho,t)dx.$$

Let us consider the integral

$$\int_0^\infty e^{-\rho y}H(x,y,t)dy \quad (3)$$

Now In order to prove that the (2.3) converges at $\rho=\rho_0$

Consider

$$\beta(x,y,t)=\int_0^y e^{-\rho y}H(x,u,t)du, 0<y<\infty \quad (4)$$

Where $\beta(x,0,t)=0$

Now $\lim_{y\to\infty}\beta(x,y,t)$ exist

Since the integral $\int_0^\infty e^{-\rho y}H(x,y,t)dy$ converges at $\rho=\rho_0$.

By using Fundamental theorem of integral calculus, we have equation (4)

$$\beta_y(x,y,t)=H(x,u,t)e^{-\rho_0 y}$$

We take ϵ and η, such that $0<\epsilon<\eta$, then

$$\int_\epsilon^\eta e^{-\rho y}H(x,y,t)dy=\int_\epsilon^\eta e^{-\rho y}\beta_y(x,y,t)e^{\rho_0 y}dy$$

$$=\int_\epsilon^\eta e^{-(\rho-\rho_0)y}\beta_y(x,y,t)dy$$

Here, we apply the Integrating by parts theorem, we have

$$=[e^{-(\rho-\rho_0)y}\beta(x,y,t)]_\epsilon^\eta-\int_\epsilon^\eta[-(\rho-\rho_0)e^{-(\rho-\rho_0)y}\beta(x,y,t)]dy$$

$$=\{[e^{-(\rho-\rho_0)\eta}\beta(x,\eta,t)]-[e^{-(\rho-\rho_0)\epsilon}\beta(x,\epsilon,t)]\}$$

$$+(\rho-\rho_0)\int_\epsilon^\eta[e^{-(\rho-\rho_0)y}\beta(x,y,t)]dy$$

If we take $\epsilon \to 0$,then second term vanishes

$$= [e^{-(\rho-\rho_0)\eta}\beta(x,\eta,t)] + (\rho-\rho_0)\int_0^{\eta}[e^{-(\rho-\rho_0)y}\beta(x,y,t)]dy$$

Let $\eta \to \infty$ and if $\rho < \rho_0$ then the First term become vanishes.

We have

$$\int_0^{\infty} e^{-\rho y} H(x,y,t)dy = (\rho-\rho_0)\int_0^{\eta}[e^{-(\rho-\rho_0)y}\beta(x,y,t)]dy \quad (5)$$

Since

$$\lim_{y\to\infty} e^{-(\rho-\rho_0)y}\beta(x,y,t) = \lim_{y\to\infty}\frac{1}{e^{(\rho-\rho_0)y}}\lim_{y\to\infty}\beta(x,y,t) = 0$$

This shows that the Equation (5) converges if $\rho < \rho_0$

Similarly, we can show that the integrals

$$\int_0^{\infty} e^{-\sigma x} H(x,y,t)dx \quad (6)$$

&

$$\int_0^{\infty} e^{-\delta t} H(x,y,t)dt \quad (7)$$

both converges at $\sigma < \sigma_0$ and $\delta < \delta_0$ respectively

Making use of expressions (5),(6), and equation (7) , Expression (2) converges at $\sigma < \sigma_0$, $\rho < \rho_0$ and $\delta < \delta_0$.

Theorem 2.4 *If*

$$L_x L_y L_t(f(x,y,t) = F(\sigma,\rho,\delta)$$

and

$$G(x,y,t) = \int_0^x\int_0^y\int_0^t f(u,v,w)dudvdw \quad (8)$$

then

$$L_x L_y L_t \int_0^x\int_0^y\int_0^t f(u,v,w)dudvdw = \frac{1}{\sigma\rho\delta}F(\sigma,\rho,\delta)$$

(9)

Proof.We consider

$$h(x,y,t) = \int_0^y f(x,y,t)dv$$

By the use of Fundamental theorem of Integral calculus, we have

$$h_y(x,y,t) = f(x,y,t) \quad (10)$$

With $h(x,0,t) = 0$ Applying the Double Laplace transform on both sides of expression (10) we have

$$H(\sigma,\rho,\delta) = \frac{1}{\rho}F(\sigma,\rho,\delta) \quad (11)$$

From expression (2.8) , we have

$$r(x,y,t) = \int_0^t h(x,y,w)dw \quad (12)$$

with $r(x,y,0) = 0$ Again by using Fundamental theorem of calculus, we have

$$r_t(x,y,t) = h(x,y,t)$$

Making use of the Double Laplace transform, we obtain

$$R(\sigma,\rho,\delta) = \frac{1}{\delta}H(\sigma,\rho,\delta) \quad (13)$$

Similarly if we take

$$g(x,y,t) = \int_0^x r(u,y,t)du$$

with $g(0,y,t) = 0$ we obtain

$$G(\sigma,\rho,\delta) = \frac{1}{\sigma} R(\sigma,\rho,\delta) \quad (14)$$

Making use of the expressions (11),(13) and (14) , we obtain equation (9).

3. Application

Example 3.1 *Consider the Volterra-integro partial differential equation as*

$$\frac{\partial V(x,y,t)}{\partial x} + \frac{\partial V(x,y,t)}{\partial y} + \frac{\partial V(x,y,t)}{\partial t} = -x^2y^2t^2 + 4xy + 4xt + 4yt + 2\int_0^x\int_0^y\int_0^t V(p,q,r)dpdqdr \quad (15)$$

Subject to the initial conditions

$$V(x,,y,0) = 0, V(x,0,t) = 0, V(0,y,t) = 0 \quad (16)$$

Solution:Appling the formula of triple Laplace transform on both sides of equation (15)we have

$$L\left[\frac{\partial V(x,y,t)}{\partial x}\right] + L\left[\frac{\partial V(x,y,t)}{\partial y}\right] + L\left[\frac{\partial V(x,y,t)}{\partial t}\right]$$
$$= L[-x^2y^2t^2] + L[4xy] + L[4xt] + L[4yt] + \{L\left[2\int_0^x\int_0^y\int_0^t V(k,r,s)dkdrds\right]\}$$

Therefore,

$$[\rho v(\rho,s,\delta) - v(0,s,\delta)] + [sv(\rho,s,\delta) - v(\rho,0,\delta)] + [\delta v(\rho,s,\delta) - v(\rho,s,0)]$$

$$= -\left[\frac{2}{\rho^3}\right]\left[\frac{2}{s^3}\right]\left[\frac{2}{\delta^3}\right] + 4\left[\frac{1}{\rho^2}\right]\left[\frac{1}{s^2}\right]\left[\frac{1}{\delta^2}\right] + 4\left[\frac{1}{\rho^2}\right]\left[\frac{1}{s}\right]\left[\frac{1}{\delta^2}\right] + 4\left[\frac{1}{\rho}\right]\left[\frac{1}{s^2}\right]\left[\frac{1}{\delta^2}\right] + \frac{2}{\rho s\delta} v(\rho,s,\delta).$$

Using initial condition (16), we obtain

$$\rho v(\rho,s,\delta) + sv(\rho,s,\delta) + \delta v(\rho,s,\delta)$$
$$= -\frac{8}{\rho^3s^3\delta^3} + \left[\frac{4}{\rho^2s^2\delta}\right] + \left[\frac{4}{\rho^2s\delta^2}\right] + \left[\frac{4}{\rho s^2\delta^2}\right] + \frac{2}{\rho s\delta} v(\rho,s,\delta).$$

simplifying, we get

$$v(\rho,s,\delta) = 4\left[\frac{1}{\rho^2s^2\delta^2}\right]. \quad (17)$$

Making use of the inverse triple Laplace transform on both sides of the expression (17) we have

$$V(x,y,t) = 4xyt$$

This gives an exact analytical solution to (15), in which the same solution obtained by other existing methods.

4. Conclusions

This paper intends to show the applicability of the Triple Laplace transform to obtain the solution for Volterra-integro partial differential Equation with initial conditions. It concludes that the Triple Laplace transform is very powerful, effective and efficient tool. The obtained result, by this Triple Laplace transform method is found matched with exact solution obtained by other existing methods. We also examined and verified the iterative procedure for fast convergence and using TLT method, solution for VIPDE has been found.

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Authors' Contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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