Research Article

Asymptotic Formulas for Weight Numbers of the Boundary Problem differential operator on a Star-shaped Graph

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Abstract: In this article the boundary value problem differential operator on the graph of a special structure is considered. The graph has edges, joined at one common vertex, and vertices of degree 1. The boundary value problem is set by the Sturm – Liouville differential expression with real-valued potentials, the Dirichlet boundary conditions, and the standard matching conditions. This problem has a countable set of eigenvalues. We consider the so-called weight numbers, being the residues of the diagonal elements of the Weyl matrix in the eigenvalues. These elements are monomorphic functions with simple poles which can be only the eigenvalues. We note that the considered weight numbers generalize the weight numbers of differential operators on a finite interval, equal to the reciprocals to the squared norms of eigenfunctions. These numbers together with the eigenvalues play a role of spectral data for unique reconstruction of operators. We obtain asymptotic formulas for the weight numbers using the contour integration, and in the case of the asymptotically close eigenvalues the formulas are got for the sums. The formulas can be used for the analysis of inverse spectral problems on the graphs..

Keywords: boundary problem, asymptotic formulas, weight numbers, star-shaped graph

1. Introduction

We cconsider the graph Γ which consists of m edges $e_j,\ m\geq 2,\ j=\overline{1,m},$ joined at a common vertex. We let the graph Γ be parameterized so that $x_j\in[0,\pi]$ where the parameter x_j corresponds to the edge e_j, the parameter $x_j=0$ in the boundary vertex and $x_j=\pi$ in the common vertex, $j=\overline{1,m}$. We call Γ a star-shaped graph.

A function on the graph is a vector function

$$y(x)=y_j(x_j),\quad j=1,\ldots,m$$

Where the components $y_j(x_j)$ are functions on the edges e_j correspondingly $y_j(x_j)\in L_2^2[0,\pi],\ j=1,\ldots,m$.

We denote by g' differentiation of the function g with respect to the first argument. Consider the differential expression

$$Ly:=-y_j''(x_j)+q_j(x_j)y_j(x_j),\qquad j=1,\ldots,m \tag{1}$$

Then the boundary value problem **differential operator** on the graph can be written as follows:

$$Ly=\lambda y(x) \tag{2}$$

$$y_1(0)=y_2(0)\ldots=y_m(0)=0 \tag{3}$$

$$-y_1'(\pi)=\sum_{j=2}^{m-1}y_j'(\pi) \tag{4}$$

$$y_1(\pi)=y_2(\pi)=\ldots=y_m(\pi) \tag{5}$$

where λ is the spectral parameter, the equalities (3) are Dirichlet conditions, and (4)–(5) are the standard matching conditions. In (1) the functions $q_j(x_j)$ are called potentials, $q_j(x_j)\in L_2[0,\pi],\ q_j(x_j)\in\square$. The differential operator L ,given by the differential expression (1) and the conditions (3)–(5), is self-adjoin in the corresponding Hilbert space (see [1] for details).Since the differential operators on the graphs have applications in physics, chemistry, nano-technology, they are studied actively (see [2,3]). In the article we obtain asymptotic formulas for weight numbers of the problem (2)–(5) which generalize the weight numbers on a finite interval [4,Chapter1]. Those asymptotic formulas can be applied for studying of inverse spectral problems for differential operators on graphs. Weight numbers together with eigenvalues have been used for reconstruction of the potentials of the Sturm–Liouville operators on graphs, e.g.,in[5,6]. The difficult case is when the eigenvalues are asymptotically close though not multiple. The asymptotic formulas are got by using the integration over the contours, containing the asymptotically close eigenvalues, in the plain of the spectral parameter. Thus, the asymptotic formulas are obtained for the sums of the weight numbers, as it has been done in [7] for the weight matrices for the matrix Sturm – Liouville operator.

2. Objectives of this research

The main purpose of this research paper is to obtain asymptotic formulas for weight numbers of the boundary problem differential operator on a Star-shaped graph.

3. Methodology

A descriptive research project to focus and identify the effect of differential equations on asymptotic formulas for weight numbers of the boundary problem differential operator on a Star-shaped graph. Books, journals, and websites have been used to advance and complete this research.

4 Elementary Basic

In this section we introduce a characteristic function of the operator L, the zeros of which coincide with the eigenvalues. We also provide auxiliary results from [8, 9], related to the eigenvalues of L .

The conditions (4)–(5) can be written as follows:

$$V(y) := Hy'(\pi) + hy'(\pi) = 0$$

where H and h are $m \times m$ matrices :

$$H = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$$

For each fixed $j = \overline{1,m}$ let $S_j(x,\lambda)$ and $C_j(x,\lambda)$ be the solutions of the Cauchy problems

$$\begin{aligned} -S_j''(x,\lambda) + q_j(x)S_j(x,\lambda) = \lambda S_j(x,\lambda), &\quad S_j(0,\lambda) = S_j'(0,\lambda) - 1 = 0, \\ -C_j''(x,\lambda) + q_j(x)C_j(x,\lambda) = \lambda C_j(x,\lambda), &\quad C_j(0,\lambda) - 1 = C_j'(0,\lambda) = 0. \end{aligned}$$

The functions $S_j(x,\lambda)$, $C_j(x,\lambda)$ satisfy the volterra integral equations

$$S_j(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}} q_j(t)S_j(t,\lambda)dt \tag{6}$$

$$C_j(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}} q_j(t)C_j(t,\lambda)dt \tag{7}$$

Put $\tau := \operatorname{Im}\lambda$ we can obtain the following asymptotic formulas from (6), (7) as $\sqrt{\lambda} \to \infty$:

$$S_j(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\int_0^x\frac{\sin\sqrt{\lambda}(x-t)}{\lambda}\operatorname{Sin}\sqrt{\lambda}t\,q_j(t)dt+$$
$$+\int_0^x\int_0^t\frac{\sin\sqrt{\lambda}(x-t)q_j(t)}{\lambda\sqrt{\lambda}}\sin\sqrt{\lambda}(t-\xi)q_j(\xi)\sin\sqrt{\lambda}\xi d\xi\,dt+O\left(\frac{e^{|\tau|x}}{\lambda^2}\right) \tag{8}$$

$$S'_j(x,\lambda)=\cos\sqrt{\lambda}x+\int_0^x\frac{\cos\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}\operatorname{Sin}\sqrt{\lambda}t\,q_j(t)dt+$$
$$+\int_0^x\int_0^t\frac{\cos\sqrt{\lambda}(x-t)q_j(t)}{\lambda}\sin\sqrt{\lambda}(t-\xi)q_j(\xi)\sin\sqrt{\lambda}\xi d\xi\,dt+O\left(\frac{e^{|\tau|x}}{\lambda\sqrt{\lambda}}\right) \tag{9}$$

$$C_j(x,\lambda)=\cos\sqrt{\lambda}x+\int_0^x\frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}\cos\sqrt{\lambda}t\,q_j(t)dt+$$
$$+\int_0^x\int_0^t\frac{\sin\sqrt{\lambda}(x-t)q_j(t)}{\lambda}\sin\sqrt{\lambda}(t-\xi)q_j(\xi)\cos\sqrt{\lambda}\xi d\xi\,dt+O\left(\frac{e^{|\tau|x}}{\lambda\sqrt{\lambda}}\right) \tag{10}$$

$$C'_j(x,\lambda)=-\sqrt{\lambda}\sin\sqrt{\lambda}x+\int_0^x\cos\sqrt{\lambda}(x-t)\cos\sqrt{\lambda}t\,q_j(t)dt+$$
$$+\int_0^x\int_0^t\frac{\sin\sqrt{\lambda}(x-t)q_j(t)}{\sqrt{\lambda}}\sin\sqrt{\lambda}(t-\xi)q_j(\xi)\cos\sqrt{\lambda}\xi d\xi\,dt+O\left(\frac{e^{|\tau|x}}{\lambda\sqrt{\lambda}}\right) \tag{11}$$

We introduce matrix solutions of equation (2): $S(\lambda)=diag\left\{S_j(x_j,\lambda)\right\},\ \ j=1,2,...,m$ and $C(\lambda)=diag\left\{C_j(x_j,\lambda)\right\},\ \ j=1,2,...,m$. Every eigenvalue of problem (2)–(5) corresponds to the zero of the following characteristic function $\Delta(\lambda)$:

$$\Delta(\lambda):=\det V(S(\lambda)) \tag{12}$$

As $S_j(\pi,\lambda)$, $S'_j(\pi,\lambda)$ are entire functions of λ, the function $\Delta(\lambda)$ is also entire. Recon-strutting the determinant in (12), we obtain

$$\Delta(\lambda)=\sum_{k=1}^{m}\left(S'_k(\pi,\lambda)\prod_{\substack{j=1\\ j\neq k}}^{m}S_j(\pi,\lambda)\right) \tag{13}$$

Lemma 1. The number λ_0 is an eigenvalue of problem (2)–(5) of multiplicity k if and only if λ_0 is a zero of characteristic function of multiplicity k. The statement of the Lemma 1 results from the self- adroitness of L and is proved with the same technique as in [7, Lemma 3]. From the self- adroitness of L it also follows that the eigenvalues of the boundary problem (2)–(5) are real.

Denote $w_j=\frac{1}{2}\int_0^{\pi}q_j(t)dt,\ f(z)=\prod_{j=1}^{m}(z-w_j).$ Let $z^{(j)},\ j=\overline{1,m-1}$ be the zeros of $f'(z),\ z^{(m)}=\sum_{j=1}^{m}\frac{w_j}{m}.$

We will mean by $\left\{\kappa_n\right\}_{n=1}^{\infty}$ different sequences from l^2. Considering these designations, the following theorem can be formulated:

Theorem 1. The operator L has a countable set of the eigenvalues. The eigenvalues can be enumerated in such way that the further formulas are satisfied:

$$\sqrt{\lambda_n^{(j)}} = n + \frac{z^{(j)} + \kappa_n}{n\pi}, \quad j = 1,2,...,m-1, \quad n \in \mathbb{N} \tag{14}$$

$$\sqrt{\lambda_n^{(m)}} = n - \frac{1}{2} + \frac{z^{(m)} + \kappa_n}{n\pi}, \quad j = 1,2,...,m-1, \quad n \in \mathbb{N} \tag{15}$$

The formulas (14), (15) with the remainders o(1) are obtained in [1]; theorem 1 is proved for real-valued potentials by V. Pivovarchick [8] (see also [9]).

Remark 1. The statement of the Theorem 1 is also correct under the conditions $q_j(x) \in \mathbb{C}$, $j = \overline{1,m},$ all $\{z^{(k)}\}_{k=1}^{m-1}$ are distinct.

5. MAIN RESULTS

In this paper we define and study weight numbers based on the Weyl matrix.

Let $\Phi(\lambda) = \left\{\phi_{jk}(x_j, \lambda)\right\}_{j,k=1}^{m}$ be the matrix solution of (2) under the conditions $\left\{\phi_{jk}(0,\lambda)\right\}_{j,k=1}^{m} = I, \ V(\Phi) = 0.$ The matrix $M(\lambda) = \left\{-\phi'_{jk}(0,\lambda)\right\}_{j,k=1}^{m}$ is called the Weyl matrix and generalize the notion of the Weyl function for differential operators on intervals (see [4]). Weyl functions and their generalizations are natural spectral characteristics, often used for reconstruction of operators. A system of $2m$ columns of the matrix solutions $C(\lambda), S(\lambda)$ is fundamental, and one can show, that

$$M(\lambda) = \left(V(S(\lambda))^{-1} V(C(\lambda))\right) \tag{16}$$

In view of (16) the elements of the matrix $M(\lambda) = \left\{M_{k,l}(\lambda)\right\}_{k,l=1}^{m}$ can be calculated as

$$M_{k,l}(\lambda) = \frac{1}{\Delta(\lambda)} \left(\prod_{\substack{j=1 \\ j\neq k}}^{m} S_j(x,\lambda),\ C_l(x,\lambda) \right)' |_{x=\pi} \tag{17}$$

The elements of the matrix $M(\lambda)$ are monomorphic functions, and their poles may be only zeros of the characteristic function $\Delta(\lambda)$.Moreover, analogously to [7, Lemma 3], we prove the following lemma:

Lemma 2. If the number λ_0 is a pole of $M_{kl}(\lambda)$, this pole is simple.

Proof. Let λ_0 be a zero of $\Delta(\lambda)$ of multiplicity b. By virtue of theorem 1 there are exactly b linearly independent eigenfunctions $\{y_j(x_j)\}_{j=1}^{b}$.corresponding to λ_0 Denote by K such invertible matrix that first b columns of $S(\lambda_0)$ K are equal to $\{y_j(x_j)\}_{j=1}^{b}$.

If $Y(\lambda) = S(\lambda)K,$ then $S(\lambda) = Y(\lambda)K^{-1},$ and $M(\lambda) = K\left[V(Y(\lambda))\right]^{-1} V(C(\lambda))$ .It is sufficient to prove that for any element of $A(\lambda)$ the number λ_0 cannot be a pole of order greater than 1, where $A(\lambda) = \left[V(Y(\lambda))\right]^{-1} V(C(\lambda)).$ If $A(\lambda) = \{A_{sl}(\lambda)\}_{s,l=1}^{m},$ then

$$A_{sl}(\lambda) = \frac{\det\left[V(Y_1(\lambda)), V(Y_2(\lambda)), ..., V(Y_{s-1}(\lambda)), V(C_l(\lambda)), V(Y_{s+1}(\lambda)), ..., V(Y_m(\lambda))\right]}{\det V(Y(\lambda))}$$

The number λ_0 is zero of the numerator of multiplicity not less than $b-1$ from that the statement of the theorem follows.

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We introduce the constants $\alpha_{jn}^{k} = \operatorname*{Res}_{\lambda=\lambda_n^{(j)}} M_{kk}(\lambda)$ which are called weight numbers. We also mean by $\{\kappa_n(z)\}_{n=1}^{\infty}$ different sequences of continuous functions such as:

$$\sum_{n=1}^{\infty} \max_{|z|\le R} \left|\kappa_n(z)\right|^2 < \infty$$

where

$$R = 2 + \max_{s=\overline{1,m}} \left|z^{(s)}\right|$$

The main results of the paper are stated in the following two theorems.

Theorem 2. Let the eigenvalues of L be enumerated as in theorem 1, $k=\overline{1,m}$ then

$$\sum_{j\in I(n)} \alpha_{jn}^{k} = \frac{2n^2}{m\pi}\left(m-1+\frac{\kappa_n}{n}\right) \tag{18}$$

$$\alpha_{ms}^{k} = \frac{(n-\frac{1}{2})^2}{m\pi}\left(2+\frac{\kappa_n}{n}\right) \tag{19}$$

Where

$$I(n) = \bigcup_{j=1}^{m-1}\left\{\min\{s : \lambda_n^{s} = \lambda_n^{(j)}\}\right\}.$$

Proof. To prove the theorem, consider $\lambda_n(z) = n + \frac{z}{n\pi}, |z| \le R$ Substituting $\lambda = \lambda_n(z)$ into (8)–(11), we obtain

$$S_j(\pi,\lambda_n^2(z)) = \frac{(-1)^n}{n\lambda_n(z)}\left(z-\tilde{\omega}_{jn}+\frac{\kappa_n(z)}{n}\right), \quad \tilde{\omega}_{jn} = \omega_j - \hat{q}_j(2n) \tag{20}$$

$$S_j'(\pi,\lambda_n^2(z)) = (-1)^n\left(1+\frac{\kappa_n(z)}{n}\right) \tag{21}$$

$$C_j(\pi,\lambda_n^2(z)) = (-1)^n\left(1+\frac{\kappa_n(z)}{n}\right) \tag{22}$$

$$C_j'(\pi,\lambda_n^2(z)) = \frac{(-1)^n\lambda_n(z)}{n}\left(z-\tilde{\omega}_{jn}+\frac{\kappa_n(z)}{n}\right), \quad \tilde{\omega}_{jn} = \omega_j - \hat{q}_j(2n) \tag{23}$$

Where $\hat{q}_j(l) = \frac{1}{2}\int_0^{\pi} q_j(t)\cos ltdt$ We substitute (20)–(23) into (13), (17) and get

$$\Delta(\lambda_n^2(z)) = \frac{(-1)^{nm}}{n^{m-1}\lambda_n^{m-1}(z)}\left(\sum_{s=1}^{m}\prod_{\substack{j=1\\ j\ne s}}^{m}(z-\tilde{\omega}_{jn})+\frac{\kappa_n(z)}{n}\right) \tag{24}$$

$$M_{kk}(\lambda_n^2(z))\Delta(\lambda_n^2(z))=\frac{(-1)^{nm}}{n^{m-2}\lambda_n^{m-2}(z)}\left(\sum_{\substack{s=1\\ s\neq k}}^{m}\prod_{\substack{j=1\\ j\neq s,\, j\neq k}}^{m}(z-\tilde{\omega}_{jn})+\frac{\kappa_n(z)}{n}\right) \tag{25}$$

Let us denote $f_n(z)=\prod_{j=1}^{m}\left(z-\tilde{\omega}_{jn}\right)$, $\delta(r)$ is the circle of center 0 and radius $r>0$.

It can be proved that $\tilde{z}_n^{(j)}=z^{(j)}+o(1), n\to\infty,$ where $\tilde{z}_n^{(j)},\ 1,2,3,...,m-1$ are the zeros of $f_n'(z)$ if $z\in\delta(R),$ then for sufficiently large $n,\ \lambda_n^2(z)$ runs across the simple

Closed contour, which surrounds $\lambda_n^{(j)},\quad j=1,2,3,...,m-1$ Integrating $M_{kk}(\lambda),$ after the substitution $\lambda=\lambda_n^2(z)$ we have

$$\sum_{l\in I(n)}\alpha_{jn}^{k}=\frac{1}{2\pi i}\int_{z\in\delta(R)}\frac{2\lambda_n(z)}{n\pi}M_{kk}(\lambda_n^2(z))dz.$$

The following formula is obtained from the previous one and (24), (25):

$$\sum_{l\in I(n)}\alpha_{jn}^{k}=\frac{1}{2\pi i}\int_{z\in\delta(R)}\frac{2\lambda_n^2(z)}{m\pi}\frac{\sum\limits_{\substack{s=1\\ s\neq k}}^{m}\prod\limits_{\substack{j=1\\ j\neq s,\, j\neq k}}^{m}\left(z-\tilde{\omega}_{jn}\right)+\frac{\kappa_n(z)}{n}}{\prod\limits_{j=1}^{m-1}\left(z-\tilde{z}_n^{(j)}\right)+\frac{\kappa_n(z)}{n}}dz \tag{26}$$

The remainder $\frac{\kappa_n(z)}{n}$ can be excluded from the denominator of (26) with Taylor expansion as $\min\limits_{|z|=R}\prod\limits_{j=1}^{m-1}\left|z-\tilde{z}_n^{(j)}\right|>1$ if n is large enough. Besides, $\lambda_n^2(z)=n^2\left(1+\frac{\kappa_n(z)}{n}\right),|z|\leq R$ after the designation

$$g_{kn}(z)=\frac{\sum\limits_{\substack{s=1\\ s\neq k}}^{m}\prod\limits_{\substack{j=1\\ j\neq s,\, j\neq k}}^{m}\left(z-\tilde{\omega}_{jn}\right)}{\prod\limits_{j=1}^{m-1}\left(z-\tilde{z}_n^{(j)}\right)}$$

We get

$$\sum_{l\in I(n)}\alpha_{jn}^{k}=\frac{2n^2}{2m\pi^2 i}\left(\int_{z\in\delta(R)}g_{kn}(z)dz+\frac{\kappa_n}{n}\right) \tag{27}$$

We note that $\delta(r)$ contains all $\tilde{z}_n^{(j)},\ j=1,2,3,...,m-1$ for $r\geq R$ and large n. Thus,

$$\int_{z\in\delta(R)}g_{kn}(z)dz=\int_{z\in\delta(r)}g_{kn}(z)dz$$

the numerator of the fraction $g_{kn}(z)$ is a polynomial of degree $m-2$ with leading coefficient $m-1$, and its denominator is a polynomial of degree $m-1$ with leading coefficient 1. For $z\in\delta(r)$ there is the equality

$$g_{kn}(z)=\frac{m-1}{z}+O(r^{-2}),$$

and

$$\frac{1}{2\pi i}\int_{z\in\delta(r)} g_{kn}(z)=m-1+O(r^{-1}).$$

As $r\to\infty$ we obtain (18). Formula (19) is proved analogously.

Theorem 3. Let $z^{(s)}$ be a zero of $f'(z)$ of multiplicity $b(s)>0,\ 1\le p\le m$ Denote $N(s)=\left\{1\le j<m: z^{(s)}\ne z^{(j)}\right\},\ N'(s)=\left\{1\le j<m: z^{(s)}=z^{(j)}\right\},$ and $W(s)=\left\{1\le j<m: z^{(s)}\ne \omega_j\right\}$ if $p\in W(s),$ then

$$\sum_{l\in N'(s)}\alpha_{\ln}^{p}=\frac{2n^2}{m\pi}(\Omega_{ps}+\kappa_n) \tag{28}$$

Else

$$\sum_{l\in N'(s)}\alpha_{\ln}^{p}=\frac{2n^2}{m\pi}(\theta_s+\kappa_n) \tag{29}$$

Where

$$\Omega_{ps}=-\frac{\prod_{j=1}^{m}(z^{(s)}-\omega_j)}{(z^{(s)}-\omega_j)^2\prod_{j\in N(s)}(z^{(s)}-z^{(j)})},\qquad \theta_s=b(s)\frac{\prod_{j\in N(s)}(z^{(s)}-\omega_j)}{\prod_{j\in N(s)}(z^{(s)}-z^{(j)})}$$

and the product over empty set is understood as 1.

Proof. Denote by r such positive number that the circle $\left|z-z^{(s)}\right|\le r$ does not contain $z^{(j)},\ j\in N(s)$ and $\left|z^{(s)}\right|+r<R,\ r\ge C>0$ we call the circumference of that circle $\gamma(s)$ the following analogue of the formulae (27) can be proved:

$$\sum_{l\in N'(n)}\alpha_{ln}^{p}=\frac{n^2}{m^2\pi^2 i}\left(\int_{\gamma(s)}\frac{\sum_{\substack{k=1\\k\ne p}}^{m}\prod_{\substack{j=1\\j\ne k,\, j\ne p}}^{m}\left(z-\tilde{\omega}_{jn}\right)}{\prod_{j=1}^{m-1}\left(z-\tilde{z}_n^{(j)}\right)}dz+\frac{\kappa_n}{n}\right) \tag{30}$$

We designate

$$F_p(z)=\frac{\sum_{\substack{k=1\\k\ne p}}^{m}\prod_{\substack{j=1,\\j\ne k,\, j\ne p}}^{m}(z-\omega_n)}{\prod_{j=1}^{m}\left(z-z^{(j)}\right)}$$

As $\omega_j-\tilde{\omega}_{jn}=\kappa_n$ and the coefficients of $f'(z),\, f_n'(z)$ depend on $\left\{\omega_j\right\}_{j=1}^{m},\left\{\tilde{\omega}_{jn}\right\}_{j=1}^{m}$ polynomially, we have

$$\frac{\sum\limits_{\substack{k=1\\k\neq p}}^{m}\prod\limits_{\substack{j=1,\\j\neq k,j\neq p}}^{m}(z-\tilde{\omega}_{jn})}{\prod\limits_{j=1}^{m}(z-\tilde{z}_n^{(j)})}-F_p(z)=\kappa_n(z) \tag{31}$$

Where $z\in\gamma(s)$ We integrate the fraction $F_p(z)$.

First we consider $b(s)>1$. Then $z^{(s)}$ is a zero of $f(z)$ of multiplicity $b(s)+1$ (see [10, section 4.3]), and cardinality of $W(s)$ is $m-b(s)-1$ in the case when $p\in W(s)$ the function $F_p(z)$ has no pole inside $\gamma(s)$, and $\alpha_{sn}^{p}=\frac{2n^2}{m\pi}\kappa_n$, what is the same as (28). If $p\notin W(s)$ then

$$F_p(z)=\frac{b(s)\left(z-z^{(s)}\right)^{b(s)-1}\prod\limits_{j\in W(s)}\left(z-\omega_j\right)+\left(z-z^{(s)}\right)^{b(s)}\sum\limits_{k\in W(s)}\prod\limits_{\substack{j\in W(s)\\j\neq k}}\left(z-\omega_j\right)}{\left(z-z^{(s)}\right)^{b(s)}\prod\limits_{j\in N(s)}\left(z-z^{(j)}\right)}$$

and

$$\int\limits_{\gamma(s)}F_p(z)dz=\frac{b(s)\prod\limits_{j\in W(s)}\left(z^{(s)}-\omega_j\right)}{\prod\limits_{j\in N(s)}\left(z^{(s)}-z^{(j)}\right)} \tag{32}$$

Formula (29) follows from (30)–(32).

Further, let $b(s)=1$ When $z^{(s)}$ is a zero of $f(z)$, computations are the same as in the case $b(s)>1$ so we assume $f(z^{(s)})\neq 0,$ and consequently $p\in W(s)$. Rewriting $F_p(z)$ as

$$F_p(z)=\left(\frac{f(z)}{z-\omega_p}\right)'\left(f'(z)\right)^{-1}=\frac{1}{z-\omega_p}-\frac{f(z)}{(z-\omega_p)^2 f'(z)}$$

and integrating over $\gamma(s)$, we obtain (29).

6. Conclusion

As a result of the research, this article consists three parts. The first part is introduction, the second part contains preliminaries and in the third part, we present the proof of the second and third theorems and the justification of the method consider retrieving two-point boundary value problems from the finite text A set of eigenvalues of the asymptotic formula of the boundary condition coefficients and asymptotic formulas for weight numbers of the boundary problem differential operator on a Star-shaped graph. In addition, it can be said that the term weight numbers are considered. They are the remnants of the oblique elements of the Weyl matrix in certain years. These elements are famous functions with simple poles that can only have certain properties. We generalized the assumed weight numbers to the weight numbers of differential operators over a limited period of time, equal to the reciprocal of the special squared norms. These numbers, along with special features, play the role of spectral data for the unique reconstruction of operators. By using of contour integration, we obtain the unbalanced duct for the weight numbers, and in the case of closely spaced free, fourls obtained for the values. At last, formulas can be used to analyze inverse spectral in graphs.

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