On Signed Coloring of Signed Graphs

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Article History: Received: 11 January 2021; Revised: 12 February 2021; Accepted: 27 March 2021; Published online: 4 June 2021

Abstract: The chromatic number of a signed graph is the number of minimum colors required to color a signed graph such that no two adjacent vertices connected by positive edge receives the same color and two vertices connected by negative edge receives opposite color .In this paper we are finding the positive chromatic number for several graph classes. We define a mapping β+:V(G^*)→{+1,+2,....,+k},where G^* is the signed graph of the underlying graph G. The positive chromatic number is the least positive integer k such that k ∈ {+1,+2,...,+k} which holds the conditions for coloring. We use χ_+ to denote positive chromatic number and χ_0 for zero-free chromatic number

Keywords: Signed Graph, Signed coloring, Zero-free coloring, positive Chromatic number .
AMS subject classification:05C78, 05C22

1. Introduction

A Graph is defined as a triplet that includes a set of all vertices $V(S)$, set of all edges $E(S)$ along with a relation relating each edge and two vertices [1]. An edge can be directed or undirected. Here we consider edges without direction and not more than one edge connects a pair of vertices. The ordinary unsigned graph can be extended to a new concept called signed graph.

A graph S* obtained from the graph S by assigning signs to each edge of S is a signed graph.. So a graph S^* is said to be signed graph if there exist a mapping γ: $E(S^*) \rightarrow \{-1,+1\}$ .Therefore, a signed graph is defined as an ordered pair (S,γ) where S is the underlying unsigned graph and γ is the signature function that assigns signs to each edges of S. The edges which are mapped to -1 are called negative edges and the edges which are mapped to +1 are called positive edges. This concept was first introduced by Frank Harary in 1954. He introduced this concept to model individual's relations between each other. He used vertices to represent individuals and edges connecting them denoted the relation between them. The positive edge connecting individuals denotes their friendship and negative edge denotes enmity. If all the edges are mapped to +1 then it is called an all positive signed graph which is the same as the ordinary unsigned graph and if all the edges are mapped to −1 then it is called an all negative signed graph. If all the edges of a signed graph is mapped to the same sign then such signed graphs are called homogeneous otherwise called heterogeneous.

A signed graph is said to be balanced if the product of the edge sign around every cycle is positive. So a cycle is balanced if it has an even number of negative edges. Frank Harary [2] concluded that a signed graph S* is said to be balanced if and only if the vertex set $V(S^*)$ can be divided into two subsets such that vertices in the same subset are connected by positive edges and the vertices in different subsets are connected by negative edges.

The idea of signed coloring a signed graph was first introduced by Thomas Zaslavsky in 1981.The basic idea behind the coloring of signed graph is that no two adjacent vertices which are connected by positive edge receive the same color and no two adjacent vertices which are connected by negative edges may receive same color but not opposite colors. Thomas Zaslavsky defined signed coloring as a mapping β: $V(S^*) \rightarrow [-k,+k]=\{-k,-k-1,......,0,.....k-1,+k\}$.But there is a disadvantage in Zaslavsky's definition that while calculating the chromatic number he used to count only the absolute values of the color,in other words he consider 1 and -1 as a single color. So Raspaud [6] defined a set M_n such that $M_n=\{\pm1,\pm2,......,\pm k\}$; if $n=2k$ and $M_n=\{0,\pm1,\pm2,......,\pm k\}$; if $n=2k$. If we do not use the color 0 then it will become a zero-free coloring. The mapping β is said to be proper coloring if it satisfies the condition $k(u) \neq \gamma(e)k(v)$ where u and v are the end vertices of e and k is the assigned color. The chromatic number of a signed graph is the least number of colors required to color a signed graph such that the graph satisfies the proper coloring.

Lynn Takeshita in [3] has found the chromatic number for some special class of graphs. Also he found some upper bound for chromatic number. He concluded by discussing about the practical application of signed graph. In [4] Zaslavsky outlined the generalization of voltage graphs. Zaslavsky in [5] defined a balanced decomposition number. Also he found some bounds and values for the same for some special class of graphs.In [6] Raspaud et.al investigated the chromatic number for signed planar graph and also proved the well-known Brooks Theorem for a signed graph.

Thomas Zaslavsky [7] found the value of χ_0 for some of the special signed graphs and proved that χ_0 is equal to the minimum size of the vertex partition inducing an anti-balanced subgraph of Σ.He also proved that the minimum chromatic number of a positive subgraph of a signed graph switching is equal to Σ. He also concluded that determining χ_0 is an NP complete problem.

Steffen,Eckhard and Alexander Vogel [8] discussed about the coloring concepts of signed graphs in general. In this paper they introduced several versions of coloring and their corresponding chromatic number of each signed graph.

Kardos, Frantisek and Jonathan Narboni [9] disproved the conjecture by Macajovaa et. al (2016).They conjectured that four colors are enough to color signed planar graph . Thus they generalized the four color theorem and kardos Frantisek and Jonathan Narboni disproved it.

Thomas Zaslavsky [10] proved three general theorems: the balance expansion theorem, a group of formulas for counting colouring by their magnitude or their signs and addition and deletion formulas obtained by constructing one signed graph from another through adding and removing arcs.

2.Some Basic Results

1. [3]If G^* is all positive then, $\chi(+G^*)=\chi(G)$.
2. [3]If G^* is all negative then, $\chi(-G^*)=2$.

Proposition 1. *[6] For every signed graph G^*,which does not contain any loop, $\chi(G^*)\leq 2\,\chi(G)-1$.*

Theorem 1. *[6] Suppose that G^* is a simple signed graph which is connected and also that Δ be the maximum degree of G^*.If G^* is no complete, balanced odd cycle or unbalanced even cycle then $\chi(G^*)\leq\Delta(G^*)$*

In this paper we find chromatic number for some signed graph classes and also we use only positive colors for the same. That is, we define a mapping

$$\beta^+ : V(G^*) \rightarrow \{+1,+2,....,+k\}. \qquad (1)$$

The positive chromatic number is the least positive integer k such that $k \in$N. Also it holds the conditions for coloring. We use χ_+ to denote positive chromatic number and χ_0 to denote zero-free chromatic number. We consider G^* as the signed graph of the underlying graph G.

3.Main Results

3.1.Cycle

A closed trail whose starting vertex and end vertex remains same is called a cycle .It is a 2 regular graph. A graph is said to be cyclic if it has atleast one cycle in it or otherwise acyclic.

Theorem 2. *For a balanced cycle C_n^*, $\chi_+(C_n^*) = 2$.*

Proof. From the definition it is clear that the chromatic number of an homogeneous negative cycle will be equal to 1, but by Raspaud et al's [6] research it will be equal to 2.So now we discuss the case of a heterogeneous graph. So, let C_n^* be a balanced cycle. *(Since C_n^* is balanced there will be an even number of negative edges)*. Suppose that C_n^* is an even cycle.ie, n=2k; $k \in$ N. Since C_n^* is balanced, for an even cycle there will be atmost $2k-2$ negative edges. Thus we can colour these $2k-2$ vertices which are connected by negative edges by one colour and remaining vertices with another colour. So in total we need only 2 colours to colour a balanced even cycle. Now suppose that C_n^* is an odd cycle.ie ,$n = 2k+1$; $k \in$N. Since C_n^* is balanced for an odd cycle there will be atmost $2k$ negative edges. So we can colour these $2k$ vertices which are connected by negative edges by one colour and the remaining one vertex by another colour. So in total we need only two colours to colour a balanced odd cycle. Therefore, we require atleast two colours to colour a balanced signed cycle C_n^*. □

Corollary 2.1. *For a signed C_3^* ,$\chi_+(G^*)$=2.*

3.2.Sunlet Graph

By attaching pendant edge to each vertices of a cycle graph we get a Sunlet graph S_n.

Theorem 3. *For a balanced sunlet graph S_n^* which is obtained from a balanced cycle C_n^* ,$\chi_+(S_n^*)$=2.*

Proof. Let us consider a sunlet graph S_n^* which is obtained from a balanced cycle C_n^*. Let $X = \{v_i : 1 \leq i \leq n\}$ be the vertices of a cycle and let $Y = \{u_i : 1 \leq i \leq n\}$ be the set of all pendant vertices to which the vertices v_i's are adjacent. Then all u_i's in X has a degree of 3. Then we have 3 possible cases.

case 1 :If all the adjacent vertices of v_i; i = 1,2,...,n are incident on positive edges , then the chromatic number will be same as the underlying graph.

Case 2 :If all the adjacent vertices of v_i; i = 1,2,...,n are incident on negative edges than the whole graph will become negative then we can colour the whole graph with one colour.

Case 3 :If all the adjacent vertices of v_i; i = 1,2,...,n are incident on positive edges and negative edges then we can colour those adjacent vertices of v_i which are connected by negative edges with one colour and those vertex connected by positive edge with another colour. Thus we need minimum 2 colours. □

3.3.Tadpole Graph

A tadpole graph $T_{m,n}$ is a graph which is obtained by adjoining a cycle and a path.

Theorem 4. *For a balanced tadpole graph* $T^*_{m,n}$, $\chi_+(T^*_{m,n})=2$.

Proof. Consider a balanced tadpole graph $T^*_{m,n}$ which is obtained from a balanced cycle and a path. We know that a balanced cycle is 2 colorable. So we can colour the subgraph $C^*_m$with 2 colors.

Let $X = \{v_i : 1 \leq i \leq m\}$ be the set of all vertices of cycle and $Y = \{u_i : 1 \leq i \leq n\}$ be the set of all vertices of the path.

Suppose that v_i be the vertex of cycle to which the path is attached to form a tadpole. Suppose that the vertex v_i is assigned with the color 'a' .Now if v_i is adjacent to u_i with positive edge then we have to color u_i with a color which is distinct from'a' .But we can use a color distinct from '*a'* ,which is already used in the subgraph

C^*_m. And the remaining u_is can be colored using the colors used for v_is.If v_i is adjacent to u_i with negative edge then we can color u_i with the same color '*a*' and the remaining u_i's can be colored using the colors used for v_is. So we need minimum 2 colors to color a balanced tadpole $T^*_{m,n}$.

□

Theorem 5. If a graph G^* contains an all positive C^*_n

of maximum order as a proper subgraph, then $\chi_+(G^*) = n$.

Proof. Consider a balanced graph G^* with n vertices such that it contains an all positive C^*_n of maximum order .Since G^* contains atleast one all positive C^*_n we need minimum n colors to color the subgraph C^*_n of G^*.Since G^* is balanced and is not all positive it should contain atleast two negative edges. Let $G_1 = G^* - C^*_n$ .Then G_1 can be the union of cycles and trees. In either cases we need atmost two colors to color the graph and we can select those two colors from C^*_n . But we need n colors to color the subgraph C^*_n of G.Therefore we require atleast n colors to color the graph G^*. □

Corollary 5.1. *If G^* contains an all positive C^*_3 as a unique homogeneous proper subgraph then,* $\chi_+(G^*)$=3.

3.4.Friendship Graph

A friendship graph F_n can be obtained by joining n copies of C_3 graphs. It is a planar graph with 2n + 1 vertices and 3n edges.

Theorem 6. *For a balanced friendship graph* F^*_n, $\chi_+(F^*_n)=2$.

Proof. Consider a balanced friendship graph F^*_n .We know that friendship graph is a union of copies of C^*_3's.In a friendship graph all copies of C^*_3's have a common point. Let Y ={C^*_3 :union of C^*_3's form a F^*_n }. Now consider any one C^*_3 from Y .Since we are considering balanced F^*_n the copies of C^*_3's will also be balanced. Then by corollary 2.1 a balanced F^*_3 require only 2 colours. Since C^*_3 is arbitrary it will be applicable for all $C^{*\prime}_3 s \in Y$. Therefore we need atleast 2 colours to colour a friendship graph. □

3.5.Wheel Graph

Connecting a hub vertex to all the vertices of a cycle graph we get a wheel graph W_n. It will have 2(n − 1) edges.

Theorem 7. *For a balanced wheel graph* W^*_n, $\chi_+(W^*_n)=2$.

Proof. Consider a wheel graph W^*_n which is obtained from a balanced cycle in such a way that W^*_n contains no all positive C^*_3cycle. Let $X = \{v_i : 1 \leq i \leq n\}$ be the vertices of the balanced cycle and u be the central vertex. Then all v_i's are adjacent to u. Therefore $deg(v_i) = 3, \forall v_i \in X, i = 1,2,....,n$. In wheel graph we get nC^*_3's as subgraphs. So we can also call a wheel graph as a union of nC^*_3's.Since we are considering balanced wheel graphs, then nC^*_3's which we have got from the wheel graph will also be balanced .So there will be 2 negative edges and 1 positive edge for each C_3^*. Let us consider one such C_3^* and let v_1,v_2 ∈ X. Suppose we colour the central vertex with one colour. If v_1 is incident on positive edge then we colour v_1 with a different colour. Since each C_3^* is balanced v_2 is incident on negative edges. Then we can colour v_2 with the same colour of u or with the same colour of v_1. Thus we can use two colors and the same logic will be applied to other C_3^* subgraphs also. So we need minimum 2 colours to colour a wheel graph. □

3.6.Helm Graph

By adjoining a pendant edge at each vertex of a wheel graph we get a helm graph H_n.

Theorem 8. *For a balanced helm graph H_n^* which is obtained from a balanced wheel graph , $\chi_+(H_n^*)$=2.*

Proof. Consider a helm graph H_n^* which is obtained from a balanced wheel graph such that the wheel graph has no all positive C_3^*'s.

Let x be the hub of the wheel. Let $X = \{v_i : 1 \leq i \leq n\}$ be the set of all remaining vertices of the wheel and $Y = \{u_i : 1 \leq i \leq n\}$ be the set of all pendant vertices which are adjacent to vi;$1 \leq i \leq n$. Therefore $deg(v_i) = 4, \forall v_i \in X$ and $deg(u_i) = 1, \forall u_i \in Y$ .From the above theorem we can colour a balanced wheel graph with 2 colours.

Consider one $u_i \in Y$ and one $v_i \in X$. If u_i is incident on positive edge then we can colour u_i with the same colour of x which is not adjacent to u_i and if u_i is incident on negative edge then we can colour u_i with the same colour of v_i.Therefore we can colour a helm graph H_n^* with minimum 2 colours. □

3.7.Sunflower Graph

A sunflower graph SF_k is a graph obtained by replacing each edges on the rim of W_k by a complete graph K_3.

Theorem 9. *For a balanced sunflower graph SF_k^* which is obtained from a balanced wheel,$\chi_+(SF_k^*)$=2.*

Proof. Let us consider a balanced sunflower graph SF_k^*, which is obtained from a balanced wheel graph such that there is no all positive cycles in it.

Let $X = \{v_i : 1 \leq i \leq k\}$ be the set of end vertices of the rim of W_{k^*} and $Y = \{u_i : 1 \leq i \leq k\}$ be the set of vertices adjacent to v_i and v_{i+1}.Since a balanced wheel can be colored using 2 colors. We only have to check for u_is.

For each u_i; $i = 1,2,...,k$; $deg(u_i)$=2.So we can color u_i with the same color of v_i if they share a negative edge or with different color if they share a positive edge. Therefore we need atleast 2 colors to color SF_k^*. □

Corollary 9.1. *For any signed cyclic graph G^* the zero-free chromatic number $\chi_0(G^*) \leq 3$.*

Proof. Consider a balanced signed cyclic graph G^*.We prove by induction on nodes of G^*.

First consider a balanced cyclic graph with 3 vertices. Since G^* is balanced,it should contain two negative edges and one positive edge. If we color one end vertex of the positive edge with one positive color and the other end with another negative color which is not opposite of the first color,then we can color the third vertex with the same negative color. Thus we need only two colors which is less than three.Now it is enough to prove that $\chi_0(G^*)$=3. For that add a new vertex v to C_3^* such that the newly added vertex v forms an all positive C_3^* as a proper subgraph. Since v forms an all positive C_3^* the two edges incident on v will be positive. So we need another color, which is different from first and second color, to color v. By combining these two conditions we can say that χ_0 $(G^*) \leq 3$.

Now suppose that the result is true for $n = k - 1$. Now we prove the result is true for $n=k$.

Consider a balanced cyclic graph with k vertices. Since G^* is balanced there will be atleast one cycle C_n^* with even number of negative edges. The maximum degree for any vertex in $G^*$is k−1.Choose a vertex v with degree $k - 1$ such that $v \in V(G^*)$ and

$v \notin V(C_n^*)$. Now delete the vertex v from G^*.This will result in a cyclic graph with $k-1$ vertices and we know that the result is true for

n=k − 1.Then we can color $G^* - \{v\}$ using exactly two colors. Now we remain with the vertex v. Since the degree of v is $k-1$ it will act as a central vertex. Then it will form C_3^* in G^* .Then we have two cases:

Case 1 :If v forms an all positive C_3^* as a proper subgraph then we need another color which is different from the colors assigned to the other two vertices.So we need maximum 3 colors to color G^*.

Case 2 :If v forms a balanced C_3^* which is not all positive,then we can color v using a color from the colors assigned to the end vertices of the positive edge of C_3^*.So we need only 2 colors to color G^*.

By combining these two cases we get $\chi_+(G^*)\leq 3$.

Hence the result is true for *n*=*k*. □

4.Conclusion

We have found the positive chromatic number for some basic graph classes and also for some derived classes. Also we observed that the positive chromatic number for a signed graph will always be less than or equal to zero-free chromatic number. In future one can make a study on the relationship between the number of switching and the chromatic number. Also they can find the chromatic number for some other derived classes.

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