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Research Article

Numerical Study Of The Spatial Fractional Advection-Diffusion Equation

[1]Swapnali Doley*, [2]A. Vanav Kumar, [3]Karam Ratan Singh

Department of Basic and Applied Science, NIT Arunachal Pradesh, India.
*E-mail: swapnalidoley05@gmai1.com

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Abstract: Here we discuss the advection-diffusion equation with diffusion fractional in one dimension. The study provides a Riemann-Liouville fractional derivative (RLFD) to obtain an implicit scheme of space fractional advection-diffusion equation (SFADE). The Von-Neumann techniques are illustrated for the stability and provided the implicit scheme is unconditionally stable under all condition and also a convergent. Numerical examples are illustrated the behavior of the fractional-order diffusion.

Keywords: Riemann-Liouville, fractional diffusion, advection-diffusion equation.

Introduction

Fractional derivatives and fractional integrals are the branch of mathematics, science and engineering that especially investigate of any arbitrary complex or real order which, as old as the classical calculus that we know today. In the past few decades, many researchers are attracted considerable interest through equations of fractional order derivatives, due to various applicational area in bioengineering, geophysics, geology, physics, rheology and other engineering functions in (Podlubny 1999, Metzler and Klafter 2000, Samko et al.1993). Metzler et al. (2000) the fractional derivatives-based advection or diffusion or advection-diffusion equation gives a better understanding of the complex transport dynamics and non-exponential relaxation systems. M. M et al. (2004) investigated numerical methods for the approximated solution of the FADE in one dimension with non-constant coefficients. Meerschaert and Tadjeran (2006) have proposed finite difference techniques (FDT) for left and right-sided space fractional PDEs. (D. A et al. 2000) also presented the fractional ADE to used model transport through a fluid flow of passive tracers in a porous medium in groundwater hydrology science. F. Liu et al. (2007) explored a time and space fractional dispersion-advection equation and analyzed its convergence criteria and stability condition for the numerical approximation. Advection-dispersion is the diffusion with the incorporation of the velocity field and also a diffusion due to an action of the fixed applied external force are described by the advection-dispersion/diffusion type equation. Yuste and Acedo (2005) presented a numerical method to determine the solutions of the fractional diffusion equation and compared it with the analytical solutions. Ferras et al. (2014) produced a numerical approach to determine a solution of the fractional time-based diffusion equation. By Anley and Zheng (2020) is analyzed a 1D the space fractional diffusion and fractional convection-diffusion equation related problem with spatially variable coefficients are discretized by the Fractional-Crank-Nicolson (F-C-N) scheme based on the Grunwald-Letnikov (right shifted) approximation as the extrapolation limit approach. Shen et al. (2011) focused on time and space fractional advection-diffusion equations by using Riesz-Caputo derivative fractional order and explored stability and convergence using Mathematical induction. Two methods were suggested: Richardson extrapolation and the short-memory theory to deal with this case. Tadjeran and Meerschaert (2007) made a combination of the implicit method of alternating directions with the discretization of Crank-Nicolson and the extrapolation of Richardson to solve the fractional 2-D diffusion equation. The finding shows that the system has unconditionally stable with second-order accuracy. (Zhang 2009) has implemented an implicit scheme to solve the time-space FADE and presented method has unconditionally stable and linear convergence when used a Grundwald shifted operator FDT for space fractional derivative.

The present study focuses on the following space-fractional (diffusion fractional) advection-diffusion equation (SFADE):

$$\frac{\partial u(x,t)}{\partial t} + a(x)\frac{\partial u}{\partial x} = d(x)\frac{\partial^{\alpha} u}{\partial x^{\alpha}},\ 0 \le x \le L,\ 0 \le t \le T \quad (1)$$

$$u(x,0) = \psi(x), \qquad 0 \le x \le L \quad (2)$$

$$u(0,t) = b1, u(L,t) = b2 \quad (3)$$

Where a(x), $\psi(x)$ and d(x) are advection coefficient, initial condition and diffusion coefficient respectively, b1 and b2 are the boundary conditions. The space fractional derivative $\frac{\partial^\alpha u}{\partial x^\alpha}$ is of order α $(1 < \alpha \le 2)$ (Podlubny, I. 1999).

1.1 Definition:

The derivative of the operator D_*^x with order α is written using Riemann-Liouville fractional as:

$$(D_*^x)u(x,t) = \frac{1}{\Gamma(r-\alpha)}\frac{d^r}{dx^r}\int_L^x \frac{u(t)}{(x-t)^{\alpha-r+1}}dt,\ \alpha > 0 \tag{4}$$

where $\Gamma(.)$ is Gamma functions.

1.2 Definition:

Let u be a function given on R. The Grunwald-Letnikov estimate for $1 < \alpha \le 2$ with positive order α is defined in (SG. Samko et al.1993).

$$D^\alpha u(x,t) \approx \frac{1}{h^\alpha}\sum_{k=0}^{N_x} w_k^\alpha\, u(x-kh,t) \tag{5}$$

where $w_k^\alpha = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$ are Grunwald-Letnikov coefficients which are the Taylor series expansion

$$w(z) = (1-z)^\alpha$$

We can express,

$$w_0^\alpha = 1, \ldots, w_k^\alpha = \left(1-\frac{\alpha}{k}\right)w_{k-1}^\alpha, k = 1,2,3, \ldots.$$

This paper provides an approximation using the implicit finite difference to the SFADE for a specific domain with its initial and boundary conditions. The convergence and stability of the implicit scheme are discussed along with the numerical illustrations.

2.Implicit difference approximation for SFADE

The approximation of SFADE (1) can be carried out using the finite difference techniques. Let u_i^k be the numerical approximation to $u(x_i,t_k)$. Define $t_k = k\tau,\ k = 0,1,2,\ldots.,n;\ x_i = ih,\ i = 0,1,2,\ldots.,m$. Here, $h = L/m$ is the space step and $\tau = T/n$ is the time step respectively. Now, we approximate SFADE in (1) by using an implicit finite difference approximation (IFDA) and approximated Riemann-Liouville to Grunwald-Letnikov in space fractional diffusion term as follows.

$$\frac{\left(u_i^{k+1} - u_i^k\right)}{\tau} + \frac{a(x)}{2h}\left(u_{i+1}^{k+1} - u_{i-1}^{k+1}\right) = \frac{d(x)}{h^\alpha}\sum_{j=0}^{i+1} g_j^{i+1}u_{i-j+1}^{k+1} \tag{6}$$

$$\left(u_i^{k+1} - u_i^k\right) + \tau\frac{a(x)}{2h}\left(u_{i+1}^{k+1} - u_{i-1}^{k+1}\right) = \tau\frac{d(x)}{h^\alpha}\sum_{j=0}^{i+1} g_j^{i+1}u_{i-j+1}^{k+1} \tag{7}$$

$$-\tau\frac{a(x)}{2h}u_{i-1}^{k+1} + u_i^{k+1} - \tau\frac{a(x)}{2h}u_{i+1}^{k+1} - \tau\frac{d(x)}{h^\alpha}\sum_{j=0}^{i+1} g_j^{i+1}u_{i-j+1}^{k+1} = u_i^k \tag{8}$$

when $k = 0$, using mathematical induction,

$$-\tau\frac{a(x)}{2h}u_{i-1}^{1}+u_{i}^{1}-\tau\frac{a(x)}{2h}u_{i+1}^{1}-\tau\frac{d(x)}{h^{\alpha}}\sum_{j=0}^{i+1}g_{j}^{i+1}u_{i-j+1}^{1}=u_{i}^{0} \tag{9}$$

when $k \geq 1$,

$$-\tau\frac{a(x)}{2h}u_{i-1}^{k+1}+u_{i}^{k+1}-\tau\frac{a(x)}{2h}u_{i+1}^{k+1}-\tau\frac{d(x)}{h^{\alpha}}\sum_{j=0}^{i+1}g_{j}^{i+1}u_{i-j+1}^{k+1}=u_{i}^{k} \tag{10}$$

The boundary and initial conditions are,

$$u_i^0=\psi(ih), \qquad u_0^k=b1, \qquad u_m^k=b2 \tag{11}$$

where $k=0,1,2,\ldots,n,\ \ i=0,1,2,\ldots.,m$.

2.1 Analysis of stability for the implicit scheme

Let $\tilde{u}_i^k, u_i^k$ $(i=1,2,\ldots,m-1;\ k=1,2,\ldots,n-1)$ be the solutions of difference equation (1) which satisfy the given initial conditions in equations (2) and (3). Suppose that the errors of the two solutions are defined as

$$\epsilon_i^k=\tilde{u}_i^k-u_i^k \tag{12}$$

when $k=0$, from equation (12), we can find

$$-\frac{a(x)\tau}{2h}\epsilon_{i-1}^{1}+\epsilon_{i}^{1}+\frac{a(x)\tau}{2h}\epsilon_{i+1}^{1}-\frac{d(x)\tau}{h^{\alpha}}\sum_{j=0}^{i+1}g_{j}^{\alpha}\epsilon_{i-j+1}^{1}=\epsilon_{i}^{0} \tag{13}$$

Where $i=1,2,\ldots,m-1$

If $k \geq 1$, from equation (13),

$$-\frac{a(x)\tau}{2h}\epsilon_{i-1}^{k+1}+\epsilon_{i}^{k+1}+\frac{a(x)\tau}{2h}\epsilon_{i+1}^{k+1}-\frac{d(x)\tau}{h^{\alpha}}\sum_{j=0}^{i+1}g_{j}^{\alpha}\epsilon_{i-j+1}^{k+1}=\epsilon_{i}^{k} \tag{14}$$

Theorem: The solution of (1) for the implicit scheme (8) with the boundary condition $u(0,t)=b1, u(1,t)=b2$ for all $t \geq 0$ on the finite domain $0 \leq x \leq 1$ with $0<\alpha<1$ is unconditionally stable and convergence.

Proof: The stability of the implicit scheme (8) is analysed with the help of the Von-Neumann method. From equation (8), we can get

$$-\frac{a(x)\tau}{2h}\epsilon_{i-1}^{k+1}+\epsilon_{i}^{k+1}+\frac{a(x)\tau}{2h}\epsilon_{i+1}^{k+1}-\frac{d(x)\tau}{h^{\alpha}}\sum_{j=0}^{i+1}g_{j}^{\alpha}\epsilon_{i-j+1}^{k+1}=\epsilon_{i}^{k} \tag{15}$$

when $n=0$, then

$$-\frac{a(x)\tau}{2h}\epsilon_{i-1}^{1}+\epsilon_{i}^{1}+\frac{a(x)\tau}{2h}\epsilon_{i+1}^{1}-\frac{d(x)\tau}{h^{\alpha}}\sum_{j=0}^{i+1}g_{j}^{\alpha}\epsilon_{i-j+1}^{1}=\epsilon_{i}^{0} \tag{16}$$

If $k \geq 1$, from equation (10)

$$-\frac{a(x)\tau}{2h}\epsilon_{i-1}^{k+1}+\epsilon_{i}^{k+1}+\frac{a(x)\tau}{2h}\epsilon_{i+1}^{k+1}-\frac{d(x)\tau}{h^{\alpha}}\sum_{j=0}^{i+1}g_{j}^{\alpha}\epsilon_{i-j+1}^{k+1}=\epsilon_{i}^{k} \tag{17}$$

Let $\epsilon_i^k=\rho^k\, e^{Ii\theta}$ $(I=\sqrt{-1})$, where θ is real.

$$\rho^{k+1}\left(1+\frac{a(x)\tau}{2h}(e^{I\theta}-e^{-I\theta})-\frac{d(x)\tau}{h^{\alpha}}\sum_{j=0}^{\infty}g_{j}^{(\alpha)}e^{I(1-j)\theta}\right)=\rho^{k} \tag{18}$$

Note that, $\theta=2\pi h/\omega \in [-\pi,\pi]$ is the phase angle ω is the wavelength.

We know that the binomial formula,

$$\sum_{k=0}^{\infty}w_k^{(\alpha)}e^{\pm I\theta}=\left(1-e^{\pm I\theta}\right)^{\alpha} \tag{19}$$

We can get

$$\rho(\theta) = \frac{1}{\left(1 + \frac{a(x)\tau}{2h}(e^{I\theta} - e^{-I\theta}) - \frac{d(x)\tau}{h^{\alpha}}\sum_{j=0}^{\infty} g_j^{(\alpha)} e^{I(1-j)\theta}\right)} \tag{20}$$

We determine stability for the low-frequency modes in θ =0 and the high-frequency modes with $\theta = \pi$ is associated. Obviously,

$$\rho(0) = 1 \tag{21}$$

Thus, the scheme will always be stable,

$$|\rho(\pi)| = \left|\frac{1}{(1 - d\tau)2^{\alpha}/h^{\alpha}}\right| \leq 1 \tag{22}$$

Which the difference scheme is stable.

2.2 The convergence of an IDA

Let $u(x_i, t_k)$ $(i = 1, 2, \ldots, m-1;\ k = 1, 2, \ldots, n-1)$ be the exact solutions of equation (1), (2) and (3) at mesh point (x_i, t_k). Define

$$\epsilon_i^k = u(x_i, t_k) - u_i^k,\ i = 1, 2, \ldots, m-1;\ k = 1, 2, \ldots, n$$

and $Y_k = (\epsilon_1^k, \epsilon_2^k, \ldots . \epsilon_{m-1}^k)^T$.

Using $Y^0 = 0$, substituting into (9) and (10).

when $k = 0$,

$$-\frac{a(x)\tau}{2h}\epsilon_{i-1}^1 + \epsilon_i^1 + \frac{a(x)\tau}{2h}\epsilon_{i+1}^1 - \frac{d(x)\tau}{h^{\alpha}}\sum_{j=0}^{i+1} g_j^{(\alpha)}\epsilon_{i-j+1}^1 = \epsilon_i^0 + R_{i,1} \tag{23}$$

when $k \geq 1$,

$$-\frac{a(x)\tau}{2h}\epsilon_{i-1}^{k+1} + \epsilon_i^{k+1} + \frac{a(x)\tau}{2h}\epsilon_{i+1}^{k+1} - \frac{d(x)\tau}{h^{\alpha}}\sum_{j=0}^{i+1} g_j^{(\alpha)}\epsilon_{i-j+1}^{k+1} = \epsilon_i^k + R_{i,1} \tag{24}$$

where $i = 1, 2, \ldots m-1,\ k = 0, 1, \ldots . n-1$

We use the solution,

$$\left|R_i^k\right| \leq C(\tau + h^2) \tag{25}$$

where C is the positive constant.

Theorem: Let U_i^k be the exact solution and the implicit solution (8) of SFADE (1). Then, we have the estimate

$$\left\|u_i^k - U_i^k\right\|_{\infty} \leq C(\tau + h^2) \tag{26}$$

where $\left\|u_i^k - U_i^k\right\|_{\infty} = max_{1 \leq i \leq m-1}|\epsilon_i^k|$ and constant, C.

Proof: Using mathematical induction, for $k = 0$,

Let $|Y_L^1| = |\epsilon_L^1| = max_{1 \leq i \leq m-1}|\epsilon_i^1|, k = 0, 1, \ldots n-1$

$$\begin{aligned} |Y^1|_{\infty} &= \left|-\frac{a(x)\tau}{2h}\epsilon_{i-1}^1 + \epsilon_i^1 + \frac{a(x)\tau}{2h}\epsilon_{i+1}^1 - \frac{d(x)\tau}{h^{\alpha}}\sum_{j=0}^{i+1} g_j^{(\alpha)}\epsilon_{i-j+1}^1\right| \\ &\leq -\frac{a(x)\tau}{2h}|\epsilon_{i-1}^1| + |\epsilon_i^1| + \frac{a(x)\tau}{2h}|\epsilon_{i+1}^1| - \frac{d(x)\tau}{h^{\alpha}}\sum_{j=0}^{i+1} g_j^{(\alpha)}\left|\epsilon_{i-j+1}^1\right| \leq \left|\mathrm{R}_{i,1}\right| \\ &\leq \mathrm{C}(\tau + \mathrm{h}^2) \end{aligned} \tag{27}$$

Thus, $|Y^1|_{\infty} = |\epsilon_i^1| \leq C(\tau + h^2)$

Suppose,

$$\left|\epsilon^j\right|_{\infty} \leq C(\tau + h^2), \qquad j = 1, 2, \ldots, k. \tag{28}$$

We can obtain

$$|Y^{k+1}|_\infty = \left|\epsilon_L^{k+1}\right| \le \left[\epsilon_i^{k+1} + \frac{c\tau}{2h}\left(\epsilon_{i+1}^{k+1} - \epsilon_{(i-1\}}^{k+1}\right) - \frac{d\tau}{h^\alpha}\sum_{j=0}^{i+1} g_j^{(\alpha)}\epsilon_{i-j+1}^{1}\right] \le \left[\epsilon_i^k + R_{i,k}\right] \quad (29)$$

Therefore, $|Y^{k+1}|_\infty \le C(\tau + h^2)$.

Hence, C is a constant for which

$$|Y^k|_\infty \le C(\tau + h^2) \quad (30)$$

Because $k\tau \le T$ is finite, we find the following results.

Theorem: Suppose, u_i^k be the solution. So, there is a positive constant C, such that

$$\left|u_i^k - u(x_i, t_k)\right| \le C(\tau + h), \qquad i = 1, 2, 3 \dots m-1, \qquad k = 0, 1, 2, \dots n-1. \quad (31)$$

3. Numerical description

The current section points out two examples in a given domain which analysis theoretically supports and shows that IDA scheme is unconditionally stable.

Example 1: We examine a SFADE as below

$$\frac{\partial u(x,t)}{\partial t} + a(x)\frac{\partial u}{\partial x} = d(x)\frac{\partial^\alpha u}{\partial x^\alpha} + f(x,t), \qquad 0 \le x \le L, \qquad 0 \le t \le T \quad (32)$$

with initial condition

$$u(x, 0) = x^2(1 - x), \quad (33)$$

and boundary condition,

$$u(0,t) = u(L,t) = 0 \quad (34)$$

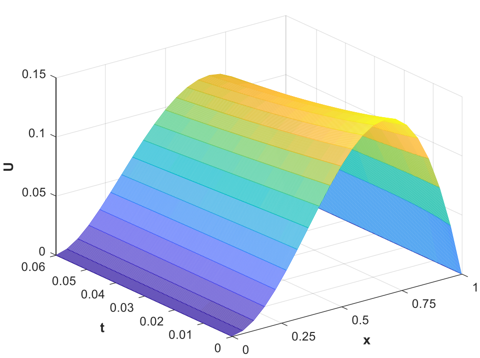
The variable coefficients of advection and diffusion are,

$$a(x) = x^{3/5}; d(x) = \Gamma(2.8)x^{3/4} \quad (35)$$

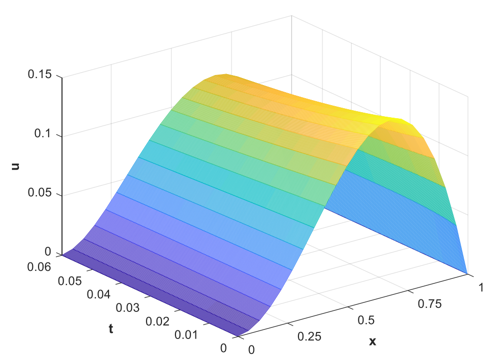
with source term is,

$$f(x,t) = \frac{2x^2(1-x)t^{1.3}}{\Gamma(2.3)} + 0.3x^{1.8}e^{-t} \quad (36)$$

The exact solution is $u(x,t) = x^2(1-x)e^{-t}$.



(a)



(b)

Fig.1. Comparison of the (a) exact and (b) numerical solution

Figure 1 denotes the validation of an exact solution and numerical solutions by using the implicit difference method. The implicit difference technique is stable and the comparison shows its accuracy. Also, the maximum absolute error observed between the numerical and exact solution are tabulated in the Table. 1.

Table 1. Maximum absolute error values ($max|U^n - u_h^n|$) for various values of α.

h	$\alpha = 1.1$	$\alpha = 1.5$	$\alpha = 1.9$
$1/20$	$6.68689189E - 03$	$4.27211449E - 03$	$4.87198913E - 03$
$1/60$	$1.03293294E - 02$	$9.42116044E - 03$	$5.86156361E - 03$
$1/100$	$1.12734595E - 02$	$1.07855788E - 02$	$7.13230204E - 03$
$1/150$	$1.18441200E - 02$	$1.14968475E - 02$	$7.42058270E - 03$
$1/200$	$1.22581907E - 02$	$1.18474280E - 02$	$1.74808372E - 02$

Example 2: In this example the characteristic of various fractional diffusion order of the SFADE is illustrated. Consider an equation,

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \frac{\partial^\alpha u}{\partial x^\alpha} \tag{37}$$

with constant coefficient and the boundary conditions, $u(0,t) = 1, u(1,t) = 0$.

This example describes the characteristics of fractional order diffusion in the advection diffusion equation. The advection on velocity, $a = 0.01, 0.1, 1$ and fractional order of the diffusion ($\alpha = 1.1, 1.6, 1.9$) affects the advection diffusion behavior as shown in fig. 2.

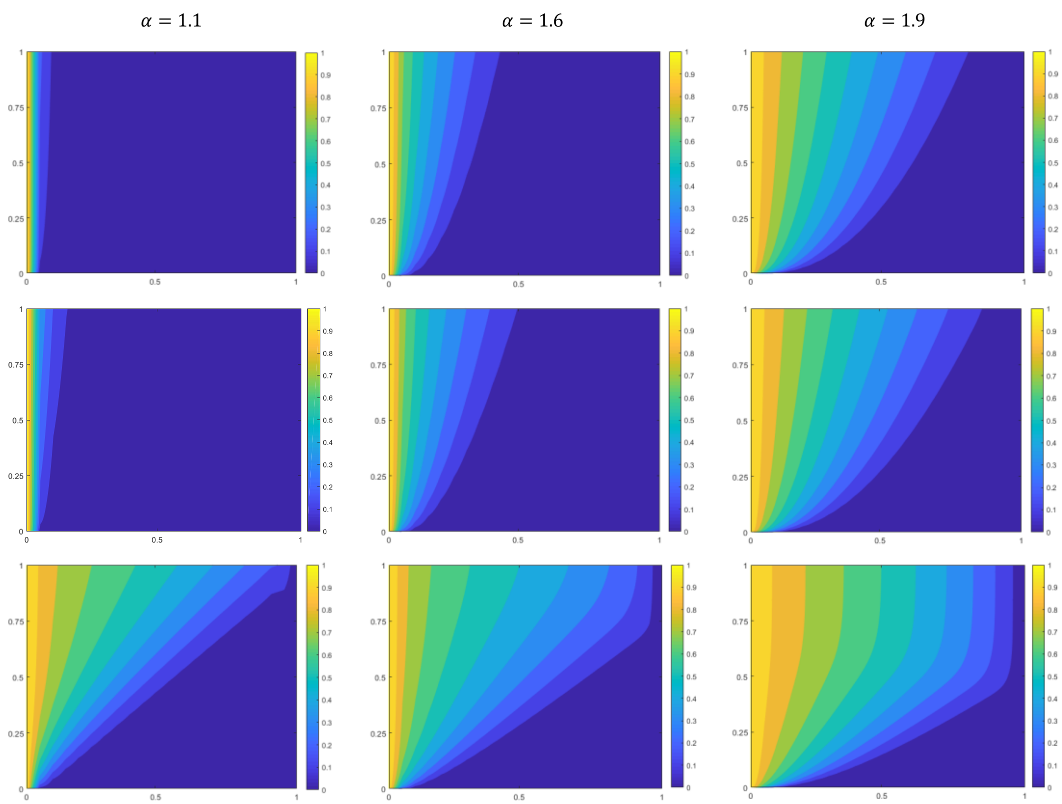


Fig. 2. Advection diffusion characteristics for various fractional order in diffusion at a=0.01 (top), a=0.1 (middle), a=1 (bottom).

4. Conclusion

The current study discussed the numerical description of the SFADE with fractional order diffusion. The diffusion fractional order derivative is explained by Riemann-Liouville derivative. The finite difference method based implicit scheme is convergent and unconditionally stable. The proposed scheme is good comparable with the exact solution.

5. Conflict of Interest

None

6. Acknowledgments
None

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