Research Article

Second order parameter uniform convergence of a finite element method for a system of 'n' partially singularly perturbed delay differential equations of reaction diffusion type

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Abstract: A boundary value problem for a second-order system of `n' partially singularly perturbed delay differential equations of reaction diffusion type is regarded in this article. This problem's solutions has boundary layers at x=0 and x=2 and inner layers at x=1. To handle the problems, a computational analysis based on a finite element method generally accessible to a piecewise-uniform Shishkin mesh is provided. It is shown that the procedure is almost second order convergent in the energy norm uniformly in the perturbation parameters. The hypothesis is supported by numerical examples.

Keywords: partially singularly perturbed problems, boundary and interior layers, delay differential equations, finite element method, Shishkin mesh, parameter - uniform convergence.

1. Introduction

We consider a boundary value problems for a system of $`n'$ partially singularly perturbed delay differential equations of reaction-diffusion type in this article. We developed a numerical method that resolves not only the normal boundary layers but also the interior layers caused by the delay terms, using a finite element method on a suitable Shishkin mesh.

The self-adjoint two-point boundary value problem that corresponds is

$$-E\,\vec{u}''(x) + A(x)\vec{u}(x) + B(x)\vec{u}(x-1) = \vec{f}(x) \quad \text{on} \quad (0,2), \tag{1.1}$$

with

$$\vec{u} = \vec{\phi} \text{ on } [-1,0] \;\; and \;\; \vec{u}(2) = \vec{l} \tag{1.2}$$

where $\vec{\phi} = (\phi_1, \phi_2, \dots, \phi_n)^T$ is sufficiently smooth on $[-1,0]$. For all $x \in [0,2]$, $\vec{u} = (u_1, u_2, \dots, u_n)^T$ and $\vec{f} = (f_1, f_2, \dots, f_n)^T$. E and $A(x)$ are $n \times n$ matrices, $E = \text{diag}(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)$ with $0 < \varepsilon_i \le 1$ for all $i = 1, \dots, n$.

The parameters $\varepsilon_i, i = 1, \dots, m$ are assumed to be distinct and, for convenience, to have the ordering $\varepsilon_1 < \cdots < \varepsilon_m < \varepsilon_{m+1} = \cdots = \varepsilon_n = 1$.

For all $x \in \overline{\Omega}$, it is assumed that the components $a_{ij}(x)$ of $A(x)$ and $b_i(x)$ of $B(x)$ satisfy the inequalities

$$a_{ii}(x) > \sum_{\substack{j \ne i \\ j=1}}^{n} \left|a_{ij}(x) + b_i(x)\right| \; for\; 1 \le i \le n \; and \;\; a_{ij}(x), b_i(x) \le 0 \;\; for \;\; i \ne j \tag{1.3}$$

and, for some α,

$$0 < \alpha < \min_{\substack{x \in [0,2] \\ 1 \le i \le n}} \sum_{j=1}^{n} \left|a_{ij}(x) + b_i(x)\right|. \tag{1.4}$$

It is assumed that $a_{ij}, b_i, f_i \in C^{(2)}(\overline{\Omega})$, for $i, j = 1, \dots, n$. Then (1.1) has a solution $\vec{u} \in C(\overline{\Omega}) \cap C^{(1)}(\Omega) \cap C^{(4)}(\Omega^- \cup \Omega^+)$.

It is also assumed that

$$\sqrt{\varepsilon_m} \le \frac{\sqrt{\alpha}}{6}. \tag{1.5}$$

The problem can also be rewritten in the form

$$\overrightarrow{L_1}\,\vec{u} = -E\,\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x) - B(x)\vec{\phi}(x-1) = \vec{g}(x) \quad \text{on} \quad \Omega^- = (0,1) \tag{1.6}$$

$$\overrightarrow{L_2}\,\vec{u} = -E\,\vec{u}''(x) + A(x)\vec{u}(x) + B(x)\vec{u}(x-1) = \vec{f}(x) \quad \text{on} \quad \Omega^+ = (1,2) \tag{1.7}$$

$$\vec{u}(0) = \vec{\phi}(0), \;\; \vec{u}(2) = \vec{l}, \qquad \vec{u}(1-) = \vec{u}(1+) \quad \text{and} \quad \vec{u}'(1-) = \vec{u}'(1+). \tag{1.8}$$

The finite element method has been analysed. Let V represent a given Hilbert space with a norm of $\|.\|_V$ and scalar product $(\cdot,\cdot)$. V is usually a subspace of the Sobolev space $H^1(\Omega)$.

Consider the weak formulation, find $\vec{u} \in H_0^1(\Omega)^n$ in particular $u_i \in H_0^1(\Omega^- \cup \Omega^+)$ for $i = 1, \ldots, n$ such that

$$\beta_{1,i}\big(u_i(x), v_i(x)\big) = g_i(v_i)(x) \ \forall \ v_i(x) \in H_0^1(\Omega^-) \qquad (1.9)$$

$$g_i(v_i)(x) = \big(f_i(x), v_i(x)\big) - \Big(\big(b_i(x)\phi_i(x-1)\big), v_i(x)\Big)$$

where $\big(u_i(x), v_i(x)\big) = \int_0^1 u_i(x)v_i(x)dx$.

$$\beta_{2,i}\big(u_i(x), v_i(x)\big) = f_i(v_i)(x) \ \forall \ v_i(x) \in H_0^1(\Omega^+) \qquad (1.10)$$

$$f_i(v_i)(x) = \big(f_i(x), v_i(x)\big).$$

where $\big(u_i(x), v_i(x)\big) = \int_1^2 u_i\,(x)v_i(x)dx$.

For $i = 1, \ldots, m$,

$$\beta_{1,i}\big(u_i(x), v_i(x)\big) = -\varepsilon_i\big(u_i'(x), v_i'(x)\big) + \left(\sum_{j=1}^{n}\Big(a_{ij}(x)u_j(x)\Big), v_i(x)\right)$$

$$\beta_{2,i}\big(u_i(x), v_i(x)\big) = -\varepsilon_i\big(u_i'(x), v_i'(x)\big) + \left(\sum_{j=1}^{n}\Big(a_{ij}(x)u_j(x)\Big), v_i(x)\right) + \big(b_i(x)u_i(x-1), v_i(x)\big).$$

For $i = m+1, \ldots, n$,

$$\beta_{1,i}\big(u_i(x), v_i(x)\big) = -\big(u_i'(x), v_i'(x)\big) + \left(\sum_{j=1}^{n}(a_{ij}(x)u_j(x)\,, v_i(x)\right)$$

$$\beta_{2,i}\big(u_i(x), v_i(x)\big) = -\big(u_i'(x), v_i'(x)\big) + \left(\sum_{j=1}^{n}\Big(a_{ij}(x)u_j(x)\Big), v_i(x)\right) + \big(b_i(x)u_i(x-1), v_i(x)\big).$$

$\beta_{1,i}\big(u_i(x), v_i(x)\big)$ and $\beta_{2,i}\big(u_i(x), v_i(x)\big)$ are bilinear forms on $H_0^1(\Omega^- \cup \Omega^+)^n$ and $g_i(v_i)(x)$, $f_i(v_i)(x)$, given continuous linear functionals on $H_0^1(\Omega^- \cup \Omega^+)^n$.

Lemma 1.1

Suppose that the bilinear forms $\beta_{1,i}(u_i(x), v_i(x))$ and $\beta_{2,i}\big(u_i(x), v_i(x)\big)$, $i = 1, \ldots, n$, is continuous on $H_0^1(\Omega^- \cup \Omega^+)^n$ is coercive, that

$$\big|\beta_{1,i}\big(u_i(x), v_i(x)\big)\big| \le \gamma_1 \,\|\, u_i(x) \,\|\,\|\, v_i(x) \,\| \qquad (1.11)$$

$$\beta_{1,i}\big(v_i(x), v_i(x)\big) \ge \alpha \,\|\, v_i(x) \,\|^2 \qquad (1.12)$$

$$\big|\beta_{2,i}\big(u_i(x), v_i(x)\big)\big| \le \gamma_2 \,\|\, u_i(x) \,\|\,\|\, v_i(x) \,\| \qquad (1.13)$$

$$\beta_{(2,i)}\big(v_i(x), v_i(x)\big) \ge \alpha \,\|\, v_i(x) \,\|^2 \qquad (1.14)$$

where α, γ_1 and γ_2 are constants that are indepentent of u_i and v_i. Then for any continuous linear functional $f_i(\cdot)$, the problem (1.9) and (1.10) has a unique solution.

A natural norm on $H_0^1(0,1)^n$ associated with the bilinear form $\beta_{1,i}(u_i(x), v_i(x))$ and $\beta_{2,i}\big(u_i(x), v_i(x)\big)$, $i = 1, \ldots, n$, is the energy norm

$$\|\, v_i \,\|_{\varepsilon_i}^2 = (\varepsilon_i \,\|\, v_i \,\|_1^2 + \alpha \,\|\, v_i \,\|_0^2)$$

where $\|\, v_i \,\|_1 = (v_i', v_i')^{\frac{1}{2}}$, $\|\, v_i \,\|_0 = (v_i, v_i)^{\frac{1}{2}}$ on $H_0^1(0,1)^n$

Lemma 1.2 *A bilinear functional* $\beta_{1,i}\big(u_i(x), v_i(x)\big)$ and $\beta_{2,i}\big(u_i(x), v_i(x)\big)$, $i = 1, \ldots, n$, *satisfies the coercive property with respect to*

$$\|\, v_i \,\|_{\varepsilon_i}^2 \le \beta_i(v_i, v_i)$$

Proof: For $i = 1, \ldots, m$

$$\beta_{1,i}(v_i, v_i) = -\varepsilon_i(v_i', v_i') + \big(\textstyle\sum_{j=1}^{n}\big(a_{ij}v_j\big), v_i\big)$$

$$= \varepsilon_i \parallel v_i \parallel_1^2 + \int_0^1 \left(\sum_{j=1}^{n} (a_{ij} v_j) \cdot v_i \right) dx$$
$$\geq \varepsilon_i \parallel v_i \parallel_1^2 + \alpha \parallel v_i \parallel_0^2.$$
$$\beta_{2,i}(v_i, v_i) = -\varepsilon_i(v_i', v_i') + \left(\sum_{j=1}^{n}(a_{ij}v_j), v_i\right) + \left(\mathrm{b_i u_i(x-1), v_i(x)}\right)$$
$$= \varepsilon_i \parallel v_i \parallel_1^2 + \int_1^2 \left(\sum_{j=1}^{n} (a_{ij} v_j) \cdot v_i \right) dx + \int_0^1 \left(\mathrm{b_i u_i(x-1), v_i(x)}\right) dx$$
$$\geq \varepsilon_i \parallel v_i \parallel_1^2 + \alpha \parallel v_i \parallel_0^2.$$

For $i = m+1, \dots, n$

$$\beta_{1,i}(v_i, v_i) = -(v_i', v_i') + \left(\sum_{j=1}^{n} (a_{ij} v_j), v_i \right)$$
$$= \parallel v_i \parallel_1^2 + \int_0^1 \left(\sum_{j=1}^{n} (a_{ij} v_j) \cdot v_i \right) dx$$
$$\geq \parallel v_i \parallel_1^2 + \alpha \parallel v_i \parallel_0^2.$$

$$\beta_{2,i}(v_i, v_i) = -(v_i', v_i') + \left(\sum_{j=1}^{n} (a_{ij} v_j), v_i \right) + \left(\mathrm{b_i u_i(x-1), v_i(x)}\right)$$
$$= \parallel v_i \parallel_1^2 + \int_1^2 \left(\sum_{j=1}^{n} (a_{ij} v_j) \cdot v_i \right) dx + \int_0^1 \left(\mathrm{b_i u_i(x-1), v_i(x)}\right) dx$$
$$\geq \parallel v_i \parallel_1^2 + \alpha \parallel v_i \parallel_0^2.$$

2. The Shishkin mesh

A piecewise uniform Shishkin mesh with N mesh-intervals is now constructed on $\Omega^- \cup \Omega^+$. Let $\Omega^N = \Omega^{-N} \cup \Omega^{+N}$ where $\Omega^{-N} = \{x_k\}_{k=1}^{\frac{N}{2}-1}$, $\Omega^{+N} = \{x_k\}_{k=\frac{N}{2}+1}^{N-1}$, $\overline{\Omega}^N = \{x_k\}_{k=0}^{N}$ and $\Gamma^N = \Gamma$. The mesh $\overline{\Omega}^N$ is a piecewise uniform mesh on [0,2] that was generated by dividing [0,1] into $2m + 1$ mesh-intervals as follows:

$[0, \sigma_1] \cup \dots \cup (\sigma_{m-1}, \sigma_m] \cup (\sigma_m, 1-\sigma_m] \cup (1-\sigma_m, 1-\sigma_{m-1}] \cup \dots \cup (1-\sigma_1, 1]$.

The points separating the uniform meshes are determined by the m parameters σ_r which are defined by $\sigma_0 = 0, \sigma_{m+1} = \frac{1}{2}$,

$$\sigma_m = \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon_m}}{\sqrt{\alpha}} \ln N\right\} \tag{2.1}$$

and, for $r = m-1, \dots, 1$,

$$\sigma_r = \min\left\{\frac{r\sigma_{r+1}}{r+1}, \frac{2\sqrt{\varepsilon_r}}{\sqrt{\alpha}} \ln N\right\}. \tag{2.2}$$

Clearly

$$0 < \sigma_1 < \dots < \sigma_m \leq \frac{1}{4}, \qquad \frac{3}{4} \leq 1 - \sigma_m < \dots < 1 - \sigma_1 < 1.$$

Then a uniform mesh of $\frac{N}{4}$ mesh-points is placed on the sub-interval $(\sigma_m, 1-\sigma_m]$, and a uniform mesh of $\frac{N}{8m}$ mesh-points is placed on each of the sub-intervals $(\sigma_r, \sigma_{r+1}]$ and $(1-\sigma_{r+1}, 1-\sigma_r]$, $r = 0, 1, \dots, m-1$, respectively.

The remaining was generated by dividing [1,2] into $2m + 1$ mesh-intervals as follows:

$$[1, 1+\tau_1] \cup \dots \cup (1+\tau_{m-1}, 1+\tau_m] \cup (1+\tau_m, 2-\tau_m] \cup \dots \cup (2-\tau_1, 2].$$

The points separating the uniform meshes are determined by the m parameters τ_r which are defined by $\tau_0 = 0, \tau_{m+1} = \frac{1}{2}$,

$$\tau_m = \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon_m}}{\sqrt{\alpha}} \ln N\right\} \tag{2.3}$$

and, for $r = m-1, \dots, 1$,

$$\tau_r = \min\left\{\frac{r\tau_{r+1}}{r+1}, \frac{2\sqrt{\varepsilon_r}}{\sqrt{\alpha}} \ln N\right\}. \qquad (2.4)$$

Clearly,

$$0 < \tau_1 < \cdots < \tau_m \leq \frac{1}{4}, \qquad \frac{3}{4} \leq 1 - \tau_m < \cdots < 1 - \tau_1 < 1.$$

Then a uniform mesh of $\frac{N}{4}$ mesh-points is placed on the sub-interval $(1+\tau_m, 2-\tau_m]$, and a uniform mesh of $\frac{N}{8m}$ mesh-points is placed on each of the sub-intervals $(1+\tau_r, 1+\tau_{r+1}]$ and $(2-\tau_{r+1}, 2-\tau_r]$, $r = 0,1,\dots,m-1$, respectively.

In practice, it is convenient to take

$$N = 8\, m\, \delta, \qquad \delta \geq 3, \qquad (2.5)$$

where m denotes the number of distinct singular perturbation parameters involved in the experiment (1.1). This produces a class of 2^{m+1} piecewise uniform Shishkin meshes $\overline{\Omega}^N$. When all of the parameters σ_r, $\tau_r = \frac{r}{8N}$, $r = 1,\dots,m$, are set to the left, the Shishkin mesh $\overline{\Omega}^N$ becomes a classical uniform mesh with the transformation parameters σ_r, τ_r and a scale N^{-1} from 0 to 2.

The following inequalities hold for the mesh Ω^N, $s = 1,\dots,m-1$

$$\begin{aligned}
&h_k \leq \frac{2}{N} \quad for \quad 1 \leq k \leq N\\
&h_k \geq \frac{1}{N} \quad for \quad \frac{N}{8} \leq k \leq \frac{3N}{8} \ and \ \frac{5N}{8} \leq k \leq \frac{7N}{8}\\
&h_k \leq \frac{1}{N} \quad for \quad 1 \leq k \leq \frac{N}{8} \ and \ \frac{3N}{8} \leq k \leq \frac{N}{2}\\
&h_k \leq \frac{1}{N} \quad for \quad \frac{N}{2} \leq k \leq \frac{5N}{8} \ and \ \frac{7N}{8} \leq k \leq N\\
&h_k \geq \frac{N}{8s} \quad for \quad \frac{N}{8s} \leq k \leq \frac{N}{8(s+1)} \ and \ \left(1 - \frac{N}{8(s+1)}\right) \leq k \leq \left(1 - \frac{N}{8s}\right)\\
&h_k \geq \frac{N}{8s} \quad for \quad 1 + \frac{N}{8s} \leq k \leq 1 + \frac{N}{8(s+1)} \ and \ \left(2 - \frac{N}{8(s+1)}\right) \leq k \leq \left(2 - \frac{N}{8s}\right)\\
&h_k \leq \frac{N}{8s} \quad for \quad 1 \leq k \leq \frac{N}{8(s)} \ and \ \left(1 - \frac{N}{8(s)}\right) \leq k \leq N.\\
&h_k \leq \frac{N}{8s} \quad for \quad \frac{N}{2} \leq k \leq 1 + \frac{N}{8(s)} \ and \ \left(2 - \frac{N}{8(s)}\right) \leq k \leq N.
\end{aligned} \qquad (2.6)$$

3. The discrete problem

In this segment, a numerical method for (1.9) and (1.10) are constructed using a finite element method with a suitable Shishkin mesh. Let for $i = 1,2,\dots,n$ and $k = 1,2,\dots,N\backslash\{N/2\}$, $V_{i,k} \subset H_0^1(\Omega^- \cup \Omega^+)^n$ be the space of piecewise linear functionals on $\Omega^- \cup \Omega^+$, that vanish at $x = 0, 1$ and 2.

The finite element approach is now established for the discrete two-point boundary value problem, $U_{i,k} \in V_{i,k} \subset H_0^1(\Omega^- \cup \Omega^+)$

$$\beta_{1,i}\big(U_{i,k}(x)v_{i,k}(x)\big) = g_{i,k}\big(v_{i,k}\big)(x) \qquad \forall \quad v_{i,k} \in V_{i,k} \subset H_0^1(\Omega^-) \qquad (3.1)$$

$$g_{i,k}\big(v_{i,k}\big)(x) = \big(f_{i,k}(x), v_{i,k}(x)\big) - \Big(\big(b_{i,k}(x)\phi_{i,k-1}(x)\big), v_{i,k}(x)\Big)$$

$$\beta_{2,i}\big(U_{i,k}(x)v_{i,k}(x)\big) = f\big(v_{i,k}\big)(x) \qquad \forall \quad v_{i,k} \in V_{i,k} \subset H_0^1(\Omega^+) \qquad (3.2)$$

$$f_{i,k}\big(v_{i,k}\big)(x) = \big(f_{i,k}(x), v_{i,k}(x)\big).$$

For $i = 1,\dots,m$,

$$\beta_{1,i}\left(u_{i,k}(x)v_{i,k}(x)\right) = -\varepsilon_i\left(u'_{i,k}(x), v'_{i,k}(x)\right) + \left(\sum_{j=1}^{n}\left(a_{ij}(x)u_{j,k}(x)\right), v_{i,k}(x)\right)$$

$$\beta_{2,i}\left(u_{i,k}(x), v_{i,k}(x)\right)$$

$$= -\varepsilon_i\left(u'_{i.k}(x), v'_{i,k}(x)\right) + \left(\sum_{j=1}^{n}\left(a_{ij}(x)u_{j,k}(x)\right), v_{i,k}(x)\right) + \left(b_{i,k}(x)u_{i,k-1}(x), v_{i,k}(x)\right).$$

For $i = m+1, \dots, n$,

$$\beta_{1,i}\left(u_{i,k}(x), v_{i,k}(x)\right) = -\left(u'_{i,k}(x), v'_{i,k}(x)\right) + \left(\sum_{j=1}^{n}(a_{ij}(x)u_{j,k}(x)\right), v_{i,k}(x))$$

$$\beta_{2,i}\left(u_{i,k}(x), v_{i,k}(x)\right) = -\left(u'_{i,k}(x), v'_{i,k}(x)\right) + \left(\sum_{j=1}^{n}\left(a_{ij}(x)u_{j.k}(x)\right), v_{i,k}(x)\right) + \left(b_{i,k}(x)u_{i,k-1}(x), v_{i,k}(x)\right).$$

By Lax-Migram, Lemma implies that

1. The discrete problem has a unique solution,
2. The discrete problem is stable.

From (1.4) on A implies that for arbitrary $x \in (0,2)$

$$\xi^T A\xi \geq \alpha\xi^T\xi \quad \forall \quad \xi \ on \ V^*_{i,k}$$

where $V^*_{i,k}$ is dual space for $V_{i,k}$.

Let $\{\phi_{i,k}: k = 1, \cdots, N\}$ be a basis for $V_{i,k}$, where $N = N(i,k)$ is the dimension of $V_{i,k}$. Then $U_{i,k} \in H^1_0(\Omega^-)$,

$$U_{i,k} = \sum_{k=1}^{\frac{N}{2}-1} C_{i,k}\phi_{i,k}$$

where the unknowns $C_{i,k}$ satisfiy the linear system

$$AU = B$$

with $A = \beta_{1,i}(\phi_{i,k_1}, \phi_{i,k_2}), U = C_{i,k}, B = g_{i,k}(\phi_{i,k})$.

The corresponding difference scheme is

$$\begin{pmatrix} \beta_{1,1}(\phi_{1,1}, \phi_{1,1}) & \beta_{1,1}(\phi_{1,1}, \phi_{1,2}) & \cdots & \beta_{1,1}\left(\phi_{1,1}, \phi_{n,\frac{N}{2}-1}\right) \\ \beta_{1,1}(\phi_{1,2}, \phi_{1,1}) & \beta_{1,1}(\phi_{1,2}, \phi_{1,2}) & \cdots & \beta_{1,1}\left(\phi_{1,2}, \phi_{n,\frac{N}{2}-1}\right) \\ \vdots & \vdots & & \vdots \\ \beta_{1,n}\left(\phi_{n,\frac{N}{2}-1}, \phi_{1,1}\right) & \beta_{1,n}\left(\phi_{n,\frac{N}{2}-1}, \phi_{1,2}\right) & \cdots & \beta_{1,n}\left(\phi_{n,\frac{N}{2}-1}, \phi_{n,\frac{N}{2}-1}\right) \end{pmatrix}$$

$$\begin{pmatrix} C_{1,1} \\ C_{1,2} \\ \vdots \\ C_{n,\frac{N}{2}-1} \end{pmatrix} = \begin{pmatrix} (g_1, \phi_{1,1}) \\ (g_1, \phi_{1,2}) \\ \vdots \\ \left(g_n, \phi_{n,\frac{N}{2}-1}\right) \end{pmatrix}.$$

$$For\ k = 1, \dots, N \qquad \begin{matrix} \phi_{1,k} = \phi_{2,k} = \cdots = \phi_{n,k} \\ C_{1,k} = C_{2,k} = \cdots = C_{n,k}. \end{matrix}$$

The nonzero contribution from a particular element is

$$A_{i,k} = \begin{pmatrix} \int_{x_{k-1}}^{x_k} \phi_{i,k-1}.\phi_{i,k-1}dx & \int_{x_{k-1}}^{x_k} \phi_{i,k-1}.\phi_{i,k}dx \\ \int_{x_k}^{x_{k+1}} \phi_{i,k}.\phi_{i,k}dx & \int_{x_k}^{x_{k+1}} \phi_{i,k}.\phi_{i,k+1}dx \end{pmatrix}.$$

Similarly, the local load vector is

$$B_{i,k} = \begin{pmatrix} \int_{x_k}^{x_{k+1}} g_i.\phi_{i,k} dx \\ \int_{x_k}^{x_{k+1}} g_i.\phi_{i,k+1} dx \end{pmatrix}.$$

For $U_{i,k} \in H_0^1(\Omega^+)$,

$$U_{i,k} = \sum_{k=1}^{\frac{N}{2}-1} C_{i,k}\phi_{i,k} + \sum_{k=\frac{N}{2}+1}^{N} C_{i,k}\phi_{i,k}$$

where the unknowns $C_{i,k}$ satisfiy the linear system

$$AU = B$$

with $A = \beta_{2,i}\left(\phi_{i,k_1}, \phi_{i,k_2}\right), U = C_{i,k}, B = f_{i,k}\left(\phi_{i,k}\right)$.

The corresponding difference scheme is

$$\begin{pmatrix} \beta_{2,1}\left(\phi_{1,\frac{N}{2}+1}, \phi_{1,\frac{N}{2}+1}\right) & \beta_{2,1}\left(\phi_{1,\frac{N}{2}+1}, \phi_{1,\frac{N}{2}+2}\right) & \cdots & \beta_{2,1}\left(\phi_{1,\frac{N}{2}+1}, \phi_{n,N-1}\right) \\ \beta_{2,1}\left(\phi_{1,\frac{N}{2}+2}, \phi_{1,\frac{N}{2}+1}\right) & \beta_{2,1}\left(\phi_{1,\frac{N}{2}+2}, \phi_{1,\frac{N}{2}+2}\right) & \cdots & \beta_{2,1}\left(\phi_{1,\frac{N}{2}+2}, \phi_{n,N-1}\right) \\ \vdots & \vdots & & \vdots \\ \beta_{2,n}\left(\phi_{n,N-1}, \phi_{n,\frac{N}{2}+1}\right) & \beta_{2,n}\left(\phi_{n,N-1}, \phi_{1,\frac{N}{2}+2}\right) & \cdots & \beta_{2,n}\left(\phi_{n,N-1}, \phi_{n,N-1}\right) \end{pmatrix}$$

$$\begin{pmatrix} C_{1,\frac{N}{2}+1} \\ C_{1,\frac{N}{2}+2} \\ \vdots \\ C_{n,N-1} \end{pmatrix} = \begin{pmatrix} \left(f_1, \phi_{1,\frac{N}{2}+1}\right) \\ \left(f_1, \phi_{1,\frac{N}{2}+2}\right) \\ \vdots \\ \left(f_n, \phi_{n,N-1}\right) \end{pmatrix}.$$

$For\ k = 1, \dots, N$

$$\phi_{1,k} = \phi_{2,k} = \cdots = \phi_{n,k}$$
$$C_{1,k} = C_{2,k} = \cdots = C_{n,k}.$$

The nonzero contribution from a particular element is

$$A_{i,k} = \begin{pmatrix} \int_{x_{k-1}}^{x_k} \phi_{i,k-\frac{N}{2}-1}.\phi_{i,k-\frac{N}{2}-1} + \phi_{i,k-1}.\phi_{i,k-1} dx & \int_{x_{k-1}}^{x_k} \phi_{i,k-\frac{N}{2}-1}.\phi_{i,k-\frac{N}{2}} + \phi_{i,k-1}.\phi_{i,k} dx \\ \int_{x_k}^{x_{k+1}} \phi_{i,k-\frac{N}{2}}.\phi_{i,k-\frac{N}{2}} + \phi_{i,k}.\phi_{i,k} dx & \int_{x_k}^{x_{k+1}} \phi_{i,k-\frac{N}{2}}.\phi_{i,k-\frac{N}{2}+1} + \phi_{i,k}.\phi_{i,k+1} dx \end{pmatrix}.$$

Similarly, the local load vector is

$$B_{i,k} = \begin{pmatrix} \int_{x_k}^{x_{k+1}} g_{i,k}.\phi_{i,k} + f_i.\phi_{i,k} dx \\ \int_{x_k}^{x_{k+1}} g_{i,k+1}.\phi_{i,k+1} + f_i.\phi_{i,k+1} dx \end{pmatrix}.$$

In the following lemmas, the proofs are given for Ω^- and Ω^+ separately. For Ω^-, the proofs are similar to those in [12] and for Ω^+ the proofs are derived.

4. Interpolation error bounds

Lemma 4.1. *Let $u_{i,k}^*$ be the $V_{i,k}$-interpolant of the solution $u_{i,k}$ of (1.1) on the fitted mesh Ω^N. Then*

$$\max_{i=1,\dots,n}\ \sup_{0<\varepsilon_i\le 1} \| u_{i,k}^* - u_{i,k} \|_{\Omega^N} \le C(N^{-1}lnN)^2 ,$$

where C is a constant independent of the parameters ε_i.

Proof: The estimate is obtained separately on each subinterval $\Omega_k = (x_{k-1}, x_k) \subset \Omega^- \cup \Omega^+$ $k = 1, \dots, N-1 \setminus \left\{\frac{N}{2}\right\}$ Note that for any function g on Ω_k $g_{i,k}^* = g_{i,k-1}\phi_{i,k-1} + g_{i,k}\phi_{i,k}$,

and so it is obvious that, on Ω_k,

$$\left|g_{i,k}^{*}(x)\right| \le \max_{\Omega_k}\left|g_{i,k}(x)\right| , \tag{4.1}$$

and it is easy to see that by using sufficient Taylor expansions

$$\left|g_{i,k}^{*}(x) - g_{i,k}(x)\right| \le Ch_k^2 \max_{\Omega_k}\left|g_{i,k}''(x)\right|. \tag{4.2}$$

From (4.2) and Lemma 3 in [11], on Ω_k,

$$\begin{aligned}\left|u_{i,k}^{*}(x) - u_{i,k}(x)\right| &\le Ch_k^2 \max_{\Omega_k}\left|u_{i,k}''(x)\right| \\ &\le C\frac{h_k^2}{\varepsilon_i}. \end{aligned}\tag{4.3}$$

Also, (4.3) using Lemma 6 and Lemma 7 in [11], on Ω_k,

$$\begin{aligned}\left|u_{i,k}^{*}(x) - u_{i,k}(x)\right| &= \left|v_{i,k}^{*}(x) + w_{i,k}^{*}(x) - v_{i,k}(x) - w_{i,k}(x)\right| \\ &\le \left|v_{i,k}^{*}(x) - v_{i,k}(x)\right| + \left|w_{i,k}^{*L}(x) - w_{i,k}^{L}(x)\right| + \left|w_{i,k}^{*R}(x) - w_{i,k}^{R}(x)\right| \\ &\le Ch_k^2 \max_{\Omega_k}\left|v_{i,k}''(x)\right| + Ch_k^2 \max_{\Omega_k}\left|w^{L_{i,k}''(x)}\right| + Ch_k^2 \max_{\Omega_k}\left|w_{i,k}^{R''}(x)\right|\end{aligned}$$

For $i = 1, \dots, m,\ \Omega_k \subset \Omega^-$,

$$\le C\left(\left(1 + \sum_{q=i}^{m} B_{1q}(x)\right) + \sum_{q=i}^{m}\frac{B_{1,q}^{L}(x)}{\varepsilon_q} + \sum_{q=i}^{m}\frac{B_{1,q}^{R}(x)}{\varepsilon_q}\right) \tag{4.4}$$

For $i = 1, \dots, m,\ \Omega_k \in \Omega^+$,

$$\begin{aligned}&\le C\left(\left(1 + \sum_{q=i}^{m} B_{1,q}(x)\right) + \sum_{q=i}^{m}\frac{B_{1,q}^{L}(x)}{\varepsilon_q} + \sum_{q=i}^{m}\frac{B_{1,q}^{R}(x)}{\varepsilon_q}\right) \\ &+ C\left(\left(1 + \sum_{q=i}^{m} B_{2,q}(x)\right) + \sum_{q=i}^{m}\frac{B_{2,q}^{L}(x)}{\varepsilon_q} + \sum_{q=i}^{m}\frac{B_{2,q}^{R}(x)}{\varepsilon_q}\right).\end{aligned}\tag{4.5}$$

The discussion now centres on whether $\frac{2\sqrt{\varepsilon_m}\ln N}{\sqrt{\alpha}} \ge \frac{1}{4}$ or $\frac{2\sqrt{\varepsilon_m}\ln N}{\sqrt{\alpha}} \le \frac{1}{4}$ should be used. In the first case $\frac{1}{\varepsilon_m} \le C(\ln N)^2$ and the result follows at once from (2.6) and (4.3).

In the second case $\tau_m = \frac{2\sqrt{\varepsilon_m}\ln N}{\sqrt{\alpha}}$. Suppose that k satisfies $1 + \frac{N}{8} \le k \le 1 + \frac{3N}{8}$. Then $h_k = \frac{2(1-2\tau_m)}{N}$ and therefore

$$\frac{h_k}{\varepsilon_m} = 2N^{-1}\frac{1 - 2\tau_m}{\varepsilon_m},$$

$\tau_m \le 1 - x_k$, and so

$$e^{-\frac{\sqrt{\alpha}(1-x_k)}{\sqrt{\varepsilon_m}}} \le e^{-\frac{\sqrt{\alpha}\tau_m}{\sqrt{\varepsilon_m}}} = e^{-2\ln N} = N^{-2}. \tag{4.5}$$

Using (4.5) and (2.6) in (4.5) gives the required result.

On the other hand, if k satisfies $\frac{N}{2} \le k \le \frac{5N}{8}$ and $\frac{7N}{8} \le k \le N$ and $r = m-1, \dots, 1$, then the discussion now centres on whether $2\frac{\sqrt{\varepsilon_r}\ln N}{\sqrt{\alpha}} \ge \frac{r\tau_{r+1}}{r+1}$ or $2\frac{\sqrt{\varepsilon_r}\ln N}{\sqrt{\alpha}} \le \frac{r\tau_{r+1}}{r+1}$ should be used. In the first case $\frac{1}{\varepsilon_r} \le C(\ln N)^2$ and the result follows at once from (2.6) and (4.3).

In the second case $\tau_r = 2\sqrt{\varepsilon_r}\ln N/\sqrt{\alpha}$ and for $s = 1, \dots, m-1$,

1. suppose that k satisfies $\frac{5N}{8(s+1)} \le k \le \frac{5N}{8(s)}$ and $N - \left(\frac{N}{8(s)}\right) \le k \le N - \left(\frac{N}{8(s+1)}\right)$. Then $h_k = \frac{8m(\tau_{r+1}-\tau_r)}{N}$ and $\tau_r \le 1 - x_k$ therefore

$$\frac{h_k}{\sqrt{\varepsilon_r}} = 8mN^{-1}\frac{\tau_{r+1} - \tau_r}{\varepsilon_r}\,. \tag{4.6}$$

Using (4.6) and (2.6) in (4.5) gives the required result.

2. If k satisfies $\frac{N}{2} \le k \le \frac{5N}{8(s+1)}$ and N$-\left(\frac{N}{8(s+1)}\right) \le k \le N$, then $h_k = 8m\frac{(\tau_{r+1}-\tau_r)}{N}$, and therefore

$$\frac{h_k}{\varepsilon_r} = 8mN^{-1}\frac{(\tau_{r+1}-\tau_r)}{\varepsilon_r}, \tag{4.7}$$

using (4.7) and (2.6) in (4.5) gives the required result.

For $i = m+1,\dots,n, \Omega_k \subset \Omega^-$

$$|u^*_{i,k} - u_{(i,k)}| \le C\left(1 + B_{1,m}(x) + C_1 B^L_{1,m}(x) + C_2\varepsilon_m\left(1 - B^L_{1,m}(x)\right) + C_1 B^R_{1,m}(x) + C_2\varepsilon_m\left(1 - B^R_{2,m}(x)\right)\right)$$

For $i = m+1,\dots,n, \Omega_k \subset \Omega^+$

$$\le C\left(1 + B_{1,m}(x) + C_1 B^L_{1,m}(x) + C_2\varepsilon_m\left(1 - B^L_{1,m}(x)\right) + C_1 B^R_{1,m}(x) + C_2\varepsilon_m\left(1 - B^R_{2,m}(x)\right)\right) + C\left(1 + B_{2,m}(x) + C_1 B^L_{2,m}(x) + C_2\varepsilon_m\left(1 - B^L_{2,m}(x)\right) + C_1 B^R_{2,m}(x) + C_2\varepsilon_m\left(1 - B^R_{2,m}(x)\right)\right)$$

This gives the required result.

For $k = \frac{N}{2}$, the source terms are assumed by

$$\left(\int_{x_{k-1}}^{x_k} g_i\left(\frac{N}{2}-1\right)dx + \int_{x_k}^{x_{k+1}} f_i\left(\frac{N}{2}+1\right)dx\right)/2$$

$$h_k = \frac{h_{k-1}+h_{k+1}}{2}, \quad h_{k-1} = (x_{k-2} - x_{k-1}), \quad h_{k+1} = (x_{k+2} - x_{k+1}),$$

$$h_{k-1} = \frac{8m(\sigma_{m-1}-\sigma_m)}{N}, \quad h_{k+1} = \frac{8m(\tau_2-\tau_1)}{N}$$

$$\frac{h_k}{\varepsilon_i} = \frac{h_{k+1}+h_{k-1}}{2\varepsilon_i} = \frac{4nN^{-1}\left((\sigma_{m-1}-\sigma_m)+(\tau_2-\tau_1)\right)}{\varepsilon_i}. \tag{4.8}$$

Using (4.8) and (2.6) in (4.3) gives the required result.

Lemma 4.2. *Let $u^*_{i,k}$ be the $V_{i,k}$ -interpolant of the solution $u_{i,k}$ of (1.1) on the fitted mesh Ω^N. Then*

$$\max_{i=1,\dots,n}\ \sup_{0<\varepsilon_i\le 1} \| u^*_{i,k} - u_{i,k} \|_{\varepsilon_i} \le C(N^{-1}\ln N)^2,$$

where C is a constant independent of ε_i.

Proof:

For $i = 1,\dots,m$ from the definition of the energy norm

$$\| u^*_{i,k} - u_{i,k} \|^2_{\varepsilon_i} = \varepsilon_i\left(\left(u^*_{i,k} - u_{i,k}\right)', \left(u^*_{i,k} - u_{i,k}\right)'\right) + \alpha\left(\left(u^*_{i,k-\frac{N}{2}} - u_{i,k-\frac{N}{2}}, u^*_{i,k-\frac{N}{2}} - u_{i,k-\frac{N}{2}}\right) + \left(u^*_{i,k} - u_{i,k}, u^*_{i,k} - u_{i,k}\right)\right). \tag{4.13}$$

Each term on the right is now treated separately. It is easy to see that the second term satisfies

$$\left(u^*_{i,k-\frac{N}{2}} - u_{i,k-\frac{N}{2}}, u^*_{i,k-\frac{N}{2}} - u_{i,k-\frac{N}{2}}\right) \le ||u^*_{i,k-\frac{N}{2}} - u_{i,k-\frac{N}{2}}||^2 \tag{4.14}$$

$$\left(u^*_{i,k} - u_{i,k}, u^*_{i,k} - u_{i,k}\right) \le \| u^*_{i,k} - u_{i,k} \|^2. \tag{4.15}$$

Using integration by parts and noting that $\left(u^*_{i,k} - u_{i,k}\right)(x_k) = 0$, for each k, the first term can be bounded as follows

$$\varepsilon_i\left(\left(u^*_{i,k} - u_{i,k}\right)', \left(u^*_{i,k} - u_{i,k}\right)'\right) = \varepsilon_i \sum_{k=1,k\ne\frac{N}{2}}^{N-1} \int_{x_{k-1}}^{x_k} \left({u^*_{i,k}}'(s) - u'_{i,k}(s)\right)^2 ds$$

$$= -\varepsilon_i \sum_{k=1,k\ne\frac{N}{2}}^{N-1} \int_{x_{k-1}}^{x_k} \left({u^*_{i,k}}''(s) - u''_{i,k}(s)\right)\left(u^*_{i,k}(s) - u_{i,k}(s)\right) ds$$

$$= \varepsilon_i \sum_{k=1,k\neq\frac{N}{2}}^{N-1} \int_{x_{k-1}}^{x_k} u''_{i,k}(s)\left(u^*_{i,k}(s) - u_{i,k}(s)\right) ds$$
$$= \left(\varepsilon_i\, u''_{i,k}, u^*_{i,k} - u_{i,k}\right),$$

where the fact that $u^{*''}_{i,k} = 0$ on each Ω_k has been used.

The estimate for the second derivative of the components of $u_{i,k}$ are contained in [11], using lemma 6 and lemma 7 in [11] then gives

$$\left|\left(\varepsilon_i\, u''_{i,k}, u^*_{i,k} - u_{i,k}\right)\right| \leq \| u^*_{i,k} - u_{i,k} \| \int_0^1 \varepsilon_i |u''_{i,k}| ds + \int_1^2 \varepsilon_i |u''_{i,k}| ds$$

$$\left|\left(\varepsilon_i\, u''_{i,k}, u^*_{i,k} - u_{i,k}\right)\right| \leq \| u^*_{i,k} - u_{i,k} \|$$
$$\int_0^1 \left(\varepsilon_i |v''_{i,k}| + \varepsilon_i \left|w^{L,''}_{i,k}\right| + \varepsilon_i \left|w^{R,''}_{i,k}\right|\right) ds + \int_1^2 \left(\varepsilon_i |v''_{i,k}| + \varepsilon_i \left|w^{L,''}_{i,k}\right| + \varepsilon_i \left|w^{R,''}_{i,k}\right|\right) ds$$
$$\leq C \| u^*_{i,k} - u_{i,k} \| \int_0^1 \left(\left(1 + \sum_{q=i}^n B_{1,q}(s)\right) + C \sum_{q=i}^n \frac{B^L_{1,q}(s)}{\varepsilon_q} + C \sum_{q=i}^n \frac{B^R_{1,q}(s)}{\varepsilon_q}\right) ds$$
$$+ \int_1^2 \left(\left(1 + \sum_{q=i}^n B_{1,q}(s)\right) + C \sum_{q=i}^n \frac{B^L_{1,q}(s)}{\varepsilon_q} + C \sum_{q=i}^n \frac{B^R_{1,q}(s)}{\varepsilon_q}\right) ds$$
$$+ \int_1^2 \left(\left(1 + \sum_{q=i}^n B_{2,q}(s)\right) + C \sum_{q=i}^n \frac{B^L_{2,q}(s)}{\varepsilon_q} + C \sum_{q=i}^n \frac{B^R_{2,q}(s)}{\varepsilon_q}\right) ds$$
$$\leq C \| u^*_{i,k} - u_{i,k} \|,$$

and so

$$\varepsilon_i\left(\left(u^*_{i,k} - u_{i,k}\right)', \left(u^*_{i,k} - u_{i,k}\right)'\right) \leq C \| u^*_{i,k} - u_{i,k} \|. \tag{4.16}$$

Combining $(4.13) - (4.16)$ leads to

$$\| u^*_{i,k} - u_{i,k} \|^2_{\varepsilon_i} \leq C \| u^*_{i,k} - u_{i,k} \| \left(1 + \alpha\right) \| u^*_{i,k} - u_{i,k} \|)$$

and the proof is completed using the estimate of $\| u^*_{i,k} - u_{i,k} \|$ from Lemma 4.1.

Lemma 4.3. *Let $u^*_{i,k}$ be the $V_{i,k}$ -interpolant of the solution $u_{i,k}$ of* (1.1) *on the fitted mesh Ω^N. Then*

$$\max_{i=1,\dots,n} \sup_{0<\varepsilon_i\leq 1} \| u^*_{i,k} - u_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N} \leq C\,(N^{-1}\ln N)^2.$$

Proof: Since $u^*_{i,k}(x_k) - u_{i,k}(x_k) = 0$, it follows from the definitions of the norms that

$$\| u^*_{i,k} - u_{i,k} \|^2_{\varepsilon_i,\overline{\Omega}^N} = \varepsilon_i\left(\left(u^*_{i,k} - u_{i,k}\right)', \left(u^*_{i,k} - u_{i,k}\right)'\right) \leq \| u^*_{i,k} - u_{i,k} \|^2_{\varepsilon_i}.$$

Using the estimate in Lemma 4.2 completes the proof.

5. **Interpolation error estimate**

Lemma 5.1. *Let $u_{i,k}$ be the solution of (1.1) and $U_{i,k}$ the solution of (3.1) and (3.2). Suppose that $V_{i,k} \subset H^1_0(\Omega^{+N})$. Then*

$$\max_{i=1,\dots,n} \left|\beta_{1,i}\left(U_{i,k} - u_{i,k}, v_i\right)\right| \leq C\,(N^{-1}\ln N)^2 \| v_{i,k} \|_{l^2(\Omega^{-N})},$$
$$\max_{i=1,\dots,n} \left|\beta_{2,i}\left(U_{i,k} - u_{i,k}, v_i\right)\right| \leq C\,(N^{-1}\ln N)^2 \| v_{i,k} \|_{l^2(\Omega^{+N})},$$

where the constant C is independent of ε_i.

Proof: Since v_i is in $V_{i,k} \in H^1_0(\Omega^{-N})$, the proof resembles that of Lemma 7.1 in [12].

$$\max_{i=1,\dots,n} \left|\beta_{1,i}\left(U_{i,k} - u_{i,k}, v_i\right)\right| \leq C\,(N^{-1}\ln N)^2 \| v_{i,k} \|_{l^2(\Omega^{-N})},$$

Since v_i is in $V_{i,k} \subset H_0^1(\Omega^{+N})$, it can be written in the form

$$v_i = \sum_{k=1}^{\frac{N}{2}-1} v_{i,k}\phi_{i,k} + \sum_{k=\frac{N}{2}+1}^{N-1} v_{i,k}\phi_{i,k}$$

and so

$$\beta_{2,i}\left(U_{i,k} - u_{i,k}, v_i\right) = \sum_{k=1}^{\frac{N}{2}-1} v_{i,k}\beta_{1,i}\left(U_{i,k} - u_{i,k}, \phi_{i,k}\right) + \sum_{k=\frac{N}{2}+1}^{N-1} v_{i,k}\beta_{2,i}\left(U_{i,k} - u_{i,k}, \phi_{i,k}\right). \quad (5.1)$$

Then, for each k, $1 \le k \le N-1\backslash\left\{\frac{N}{2}\right\}$, using (1.1), (3.1) and (3.2) and the fact that $\left(1, \phi_{i,k}\right)_{\Omega^N} = \left(1, \phi_{i,k-\frac{N}{2}}\right) = \frac{h_k + h_{k+1}}{2}$,

$$\begin{aligned}\beta_{2,i}\left(U_{i,k} - u_{i,k}, \phi_{i,k}\right) &= \sum_{j=1}^{n}\left(a_{ij}U_{j,k}, \phi_{i,k}\right) + \left(b_i U_{i,k-\frac{N}{2}}, \phi_{i,k}\right) - \left(\sum_{j=1}^{n}\left(a_{ij}u_{j,k}, \phi_{i,k}\right) + \left(b_i u_{i,k-\frac{N}{2}}, \phi_{i,k}\right)\right) \\ &= \sum_{j=1}^{n}\left(a_{ij}u_{j,k}(x_k), \phi_{i,k} + b_i u_{i,k-\frac{N}{2}}\left(x_{k-\frac{N}{2}}\right), \phi_{i,k}\right) - \left(\sum_{j=1}^{n}\left(a_{ij}u_{j,k}, \phi_{i,k}\right) + (b_i u_{i,k-\frac{N}{2}}, \phi_{i,k})\right) \\ &= \sum_{j=1}^{n}\left(\left(a_{ij}\left(u_{j,k}(x_k) - u_{j,k}\right), \phi_{i,k}\right) + \left(b_i\left(u_{ik-\frac{N}{2}}\left(x_{k-\frac{N}{2}}\right) - u_{i,k-\frac{N}{2}}\right), \phi_{i,k}\right)\right)\end{aligned}$$

Since

$$\left|u_{j,k}(x_k) - u_{j,k}\right| = \left|\int_x^{x_k} u'_{j,k}(s)ds\right| \le I_k,$$

where

$$I_k = \int_{x_{k-1}}^{x_{k+1}} \left|u'_{j,k}(s)\right| ds,$$

it follows from (2.6) that

$$\beta_{2,i}\left(U_{i,k} - u_{i,k}, \phi_{i,k}\right)| \le C\frac{(h_k + h_{k+1})}{2}(I_k + N^{-1}). \quad (5.2)$$

Assume for the moment that

$$I_k \le CN^{-1}\ln N. \quad (5.3)$$

Then (5.1)-(5.3) and the Cauchy-Schwarz inequality give

$$\begin{aligned}\left|\beta_{2,i}\left(U_{i,k} - u_{i,k}, v_i\right)\right| &\le CN^{-1}\ln N \sum_{k=1,k\neq\frac{N}{2}}^{N} \frac{(h_k + h_{k+1})^{\frac{1}{2}}}{2}\left|v_{i,k}\right|\frac{(h_k + h_{k+1})^{\frac{1}{2}}}{2} \\ &\le CN^{-1}\ln N \parallel v_{i,k} \parallel_{l^2(\overline{\Omega}^N)},\end{aligned}$$

as required.

It remains therefore to verify (5.3). From the estimate are contain Lemma 3 in [11], for the first derivative of the solution, it is clear that

$$I_k \le C\int_{x_{k-1}}^{x_{k+1}} \varepsilon_i^{-\frac{1}{2}}\left(\parallel \vec{u} \parallel_\Gamma + \parallel \vec{f} \parallel_\Omega\right)dx.$$

It follows that

$$I_k \le C\frac{(h_k + h_{k+1})}{2}/\sqrt{\varepsilon_i}, \quad (5.4)$$

and that

$$I_k \le C\frac{h_k + h_{k+1}}{2} + e^{-\frac{\sqrt{\alpha}(1-x_{k+1})}{\sqrt{\varepsilon_n}}}. \quad (5.5)$$

For $i = 1, \dots, m, k = \frac{N}{2} + 1, \dots, N-1$, then the discussion now centers on whether $\frac{2\sqrt{\varepsilon_m}\ln N}{\sqrt{\alpha}} \geq \frac{1}{4}$ or $\frac{2\sqrt{\varepsilon_n}\ln N}{\sqrt{\alpha}} \leq \frac{1}{4}$.

In the first case $\frac{1}{\sqrt{\varepsilon_m}} \leq C(\ln N)^2$ and the result follows at once from (2.6) and (5.5).

In the second case $\tau_m = \frac{2\sqrt{\varepsilon_m}\ln N}{\sqrt{\alpha}}$. Suppose that k satisfies $\frac{5N}{8} \leq k \leq \frac{7N}{8}$. Then $h_k = \frac{2(1-2\tau_m)}{N}$ and therefore

$$\frac{h_k}{\varepsilon_m} = 2N^{-1}\frac{(1-2\tau_m)}{\varepsilon_m},$$

$\tau_m \leq 1 - x_{k+1}$, and so

$$e^{-\frac{\sqrt{\alpha}(1-x_{k+1})}{\sqrt{\varepsilon_m}}} \leq e^{-\frac{\sqrt{\alpha}\tau_m}{\sqrt{\varepsilon_m}}} = e^{-2\ln N} = N^{-2}. \tag{5.6}$$

Using (5.6) and (2.6) in (5.5) gives the required result.

On the other hand, if k satisfies $\frac{N}{2} \leq k \leq \frac{5N}{8}$ and $\frac{7N}{8} \leq k < N$ and $r = m-1, \dots, 1$ then the argument now depends on whether $\frac{2\sqrt{\varepsilon_r}\ln N}{\sqrt{\alpha}} \geq \frac{r\tau_{r+1}}{r+1}$ or $\frac{2\sqrt{\varepsilon_r}\ln N}{\sqrt{\alpha}} \leq \frac{r\tau_{r+1}}{r+1}$.

In the first case $\frac{1}{\sqrt{\varepsilon_r}} \leq C\ln N$ and the result follows at once from (2.6) and (5.5).

In the second case $\tau_r = \frac{2\sqrt{\varepsilon_r}\ln N}{\sqrt{\alpha}}$ and for $s = 1, \dots, m-2$. \Za\Za

(1) Suppose that k satisfies $\frac{5N}{8(s+1)} \leq k \leq \frac{5N}{8(s)}$ and $N - \left(\frac{N}{8(s)}\right) \leq k \leq N - \left(\frac{N}{8(s+1)}\right)$. Then

$$h_k = 8n\frac{(\tau_{r+1} - \tau_r)}{N}$$

and $\tau_r \leq 1 - x_k$ therefore

$$\frac{h_k}{\sqrt{\varepsilon_r}} = 8mN^{-1}\frac{\tau_{r+1} - \tau_r}{\sqrt{\varepsilon_r}}. \tag{5.7}$$

Using (5.7) and (2.6) in (5.5) gives the required result.

(2) If k satisfies $\frac{N}{2} \leq k \leq \frac{N}{8(s+1)}$ and $N - \left(\frac{N}{8(s+1)}\right) \leq k < N$, then $h_k = \frac{8m(\tau_{r+1}-\tau_r)}{N}$ and therefore

$$\frac{h_k}{\sqrt{\varepsilon_r}} = 8nN^{-1}\frac{(\tau_{r+1} - \tau_r)}{\sqrt{\varepsilon_r}}, \tag{5.8}$$

Using (5.8) and (2.6) in (5.5) gives the required result.

(3) Finally, suppose that $s = 1, \dots, m,\ k = \left\{\frac{N}{8(s)}, N - \left(\frac{N}{8(s)}\right), \frac{N}{2} + \frac{N}{8(S)}, \frac{N}{2} - \left(\frac{N}{8S}\right)\right\}$. Then

$$I_k \leq \left(\int_{k-1}^{k} + \int_{k}^{k+1}\right)|u'_{i,k}|dx < I_{k-1} + I_{k+1}$$
$$\leq CN^{-1}\ln N$$

For $k = \frac{N}{2}$, the source terms are assumed by

$$\left(\int_{x_{k-1}}^{x_k} g_i\left(\frac{N}{2} - 1\right) + \int_{x_k}^{x_{k+1}} f_i\left(\frac{N}{2} + 1\right)\right)/2$$

$$h_k = (h_{k-1} + h_{k+1})/2, \qquad h_{k-1} = (x_{k-2} - x_{k-1}) \qquad \text{and}$$

$$h_{k+1} = (x_{k+1} - x_{k+2}), \qquad h_{k-1} = \frac{8m(\sigma_{m-1} - \sigma_m)}{N}, \qquad h_{k+1} = \frac{8m(\tau_2 - \tau_1)}{N}$$

$$\frac{h_k}{\varepsilon_i} = \frac{(h_{k-1} + h_{k+1})}{2\varepsilon_i} = \frac{4mN^{-1}\left((\sigma_{m-1} - \sigma_m) + (\tau_2 - \tau_1)\right)}{\varepsilon_i} \tag{5.9}$$

Using (5.9) and (2.6) in (5.5) gives the required result.

6. Discretization error

Lemma 6.1. *Let $u^*_{i,k}$ be the $V_{i,k}$-interpolant of the solution $u_{i,k}$ of (1.1) and $U_{i,k}$ the solution of (3.1) and (3.2). Then*

$$\max_{i=1,\dots,n} \| U_{i,k} - u^*_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N} \le C(N^{-1}\ln N)^2,$$

where the constant C is independent of the parameters ε_i.

Proof: From the coercivity of $\beta_{2,i}(,)$ in Lemma 1.1 and since $U_{i,k} - u^*_{i.k} \in V_{i,k}$,

$$\| U_{i,k} - u^*_{i,k} \|^2_{\varepsilon_i,\overline{\Omega}^N} \le C\,\beta_{2,i}(U_{i,k} - u^*_{i,k}, U_{i,k} - u^*_{i,k})$$
$$\le C[\beta_{2,i}(U_{i,k} - u_{i,k}, U_{i,k} - u^*_{i,k}) + \beta_{2,i}(u_{i,k} - u^*_{i,k}, U_{i,k} - u^*_{i,k})]$$

Using Lemma 5.1, with $v_i = U_{i,k} - u^*_{i,k}$, then gives

$$\| U_{i,k} - u^*_{i,k} \|^2_{\varepsilon_i,\overline{\Omega}^N} \le C\,(N^{-1}\ln N)^2 \| U_{i,k} - u^*_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N}.$$

Cancelling the common factor gives

$$\| U_{i,k} - u^*_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N} \le C(N^{-1}\ln N)^2,$$

as required.

Theorem 6.2. *Let $u_{i,k}$ be the solution of (1.1) and $U_{i,k}$ the solution of (3.1) and (3.2). Then*

$$\max_{i=1,\dots,n} \| U_{i,k} - u_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N} \le C(N^{-1}\ln N)^2,$$

where the constant C is independent of the parameters ε_i.

Proof: Since
$$\| U_{i,k} - u_{i,k} \|_{\varepsilon_i,\Omega^N} \le \| U_{i,k} - u^*_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N} + \| u^*_{i,k} - u_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N},$$

the result follows by combining Lemmas (4.1) and (6.1).

Theorem 6.3. *Let $u_{i,k}$ be the solution of (1.1) and $U_{i,k}$ the solution of (3.1) and (3.2). Then the following parameter uniform error estimate holds*

$$\max_{i=1,\dots,n}\ \sup_{0<\varepsilon_i\le 1} \| U_{i,k} - u_{i,k} \|_{\varepsilon_i,\overline{\Omega}^N} \le C(N^{-1}\ln N)^2$$

where the constant C is independent of the parameters ε_i.

Proof: Since $\tau_r \le \frac{2\sqrt{\varepsilon_r}\ln N}{\sqrt{\alpha}}$, $r = m,\dots,1$, consider k satisfies, $\frac{N}{2} \le k \le \frac{5N}{8s}$ and $N - \left(\frac{N}{8s}\right) \le k \le N$, $s = 1,\dots,m-1$ on a neighbourhood of the boundary layers.

Using the Cauchy Schwarz inequality and Theorem 6.2,

$$|(U_{i,k} - u_{i,k})(x_k)| = \left|\int_{\Omega_k} (U_{i,k} - u_{i,k})(s)ds\right|$$
$$\le \left(\frac{1}{\varepsilon_r}\int_{\Omega_k} 1^2 ds\right)^{\frac{1}{2}} \left(\varepsilon_r \int_{\Omega_k} \left|(U_{i,k} - u_{i,k})'(s)\right|^2 ds\right)^{\frac{1}{2}}$$
$$\le \sqrt{\frac{\tau_r}{\varepsilon_r}} \| U_{i,k} - u_{i,k} \|_{\varepsilon_r,\Omega^N}$$
$$\le CN^{-2}(\ln N)^2. \tag{6.1}$$

On the other hand, suppose that k satisfies $\frac{5N}{8} \le k \le \frac{7N}{8}$, outside the boundary layers, $h_k \ge \frac{1}{N}$ and so

$$|(U_{i,k} - u_{i,k})(x_k)|^2 \le N\,h_k |(U_{i,k} - u_{i,k})(x_k)|^2$$
$$\le N \sum_{k=\frac{5N}{8}}^{\frac{7N}{8}} h_k |(U_{i,k} - u_{i,k})(x_k)|^2$$
$$\le N \| U_{i,k} - u_{i,k} \|^2_{l^2(\Omega^N)}.$$

Using Theorem (6.2) then leads to

$$\| (U_{i,k} - u_{i,k})(x_k) \| \leq \| U_{i,k} - u_{i,k} \|_{l^2(\Omega^N)}$$
$$\leq CN^{-\frac{1}{2}} (\ln N)^2. \qquad (6.2)$$

For $k = \frac{N}{2}$,

$$h_{\frac{N}{2}} = \frac{\left(h_{\frac{N}{2}-1} + h_{\frac{N}{2}+1}\right)}{2}, h_{\frac{N}{2}-1} = \left(x_{\frac{N}{2}-2} - x_{\frac{N}{2}-2}\right) \text{ and } h_{\frac{N}{2}+1} = \left(x_{\frac{N}{2}+1} - x_{\frac{N}{2}+2}\right)$$

$$\left|\left(U_{i,\frac{N}{2}} - u_{i,\frac{N}{2}}\right)\left(x_{\frac{N}{2}}\right)\right|^2 \leq N\, h_{\frac{N}{2}} \left|\left(U_{i,\frac{N}{2}} - u_{i,\frac{N}{2}}\right)\left(x_{\frac{N}{2}}\right)\right|^2$$

$$\leq N \frac{\left(h_{\frac{N}{2}-1} + h_{\frac{N}{2}+1}\right)}{2} \left|\left(U_{i,\frac{N}{2}} - u_{i,\frac{N}{2}}\right)\left(x_{\frac{N}{2}}\right)\right|^2$$
$$\leq N \left|\left(U_{i,\frac{N}{2}} - u_{i,\frac{N}{2}}\right)\left(x_{\frac{N}{2}}\right)\right|^2_{l^2(\Omega^N)}$$

Using Theorem (6.2) then leads to

$$\left|\left(U_{i,\frac{N}{2}} - u_{i,\frac{N}{2}}\right)\left(x_{\frac{N}{2}}\right)\right| \leq \left\|U_{i,\frac{N}{2}} - u_{i,\frac{N}{2}}\right\|_{l^2(\Omega^N)}$$

Combining (6.3) and (6.4) completes the proof.

7. Numerical Illustrations

Example 7.1. Consider the BVP

$$-E\vec{u}''(x) + A(x)\vec{u}(x) + B(x)\vec{u}(x-1) = \vec{f}(x),\ for\ x \in (0,2),\ \vec{u}(x) = \vec{1},\ for\ x \in [-1,0],\ \vec{u}(2) = \vec{1}$$

Where $E = diag\ (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $A = \begin{pmatrix} 6 & -1 & 0 \\ -1 & 5(1+x) & -1 \\ -1 & -(1+x^2) & 6+x \end{pmatrix}, B = \begin{pmatrix} -0.5 \\ -0.5 \\ -0.5 \end{pmatrix}$

$\vec{f} = (e^x, 2, 1+x^2)^T$. For various values of $\varepsilon_1, \varepsilon_2, \varepsilon_3,\ N = 8k,\ k = 2^r,\ r = 3, \cdots, 8,\ and\ \alpha = 1.9.$

Using the general methods from [6], the $\vec{\varepsilon}$-uniform order of convergence and the $\vec{\varepsilon}$-uniform error constant are computed by applying fitted mesh method to the example 7.1. The following table outlines the conclusions.

$Values\ of\ D_\varepsilon^N,\quad D^N,\quad p^N,\quad p^*\ and\ C_{p^*}^N\ for\ \varepsilon_1 = \frac{\eta}{32},\quad \varepsilon_2 = \frac{\eta}{16},\quad \varepsilon_3 = 1.0.$

η	Number of mesh points N				
	64	128	256	512	1024
2^0	0.7544E-03	0.1717E-03	0.6677E-04	0.2797E-04	0.1303E-04
2^{-2}	0.1786E-02	0.2975E-03	0.1115E-03	0.4510E-04	0.2050E-04
2^{-4}	0.3974E-02	0.7429E-03	0.1842E-03	0.7169E-04	0.3064E-04
2^{-6}	0.8120E-02	0.1769E-02	0.3029E-03	0.1139E-03	0.4607E-04
2^{-10}	0.1492E-01	0.3948E-02	0.7378E-03	0.1837E-03	0.7132E-04
2^{-10}	0.2426E-01	0.8082E-02	0.1761E-03	0.3010E-02	0.1129E-03
2^{-12}	0.2426E-01	0.8082E-02	0.1761E-03	0.3010E-02	0.1129E-03
2^{-14}	0.2426E-01	0.8082E-02	0.1761E-03	0.3010E-02	0.1129E-03
D^N	0.2426E-01	0.8082E-02	0.1761E-03	0.3010E-02	0.1129E-03
P^N	0.1293E+01	0.1389E+01	0.1453E+01	0.1573E+01	
C_P^N	0.8233E+00	0.8053E+00	0.7898E+00	0.5031E+00	0.5032E+00
Computed order of $\vec{\varepsilon}$ uniform convergence, p^* = 1.293					
Computed $\vec{\varepsilon}$-uniform error constant, $C_{P^*}^N$ = 0:8233					

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