Parikh Fuzzy Vector for Finite Words of Rectangular Hilbert Space Filling Curve

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Article History: Received: 11 January 2021; Revised: 12 February 2021; Accepted: 27 March 2021; Published online: 28 April 2021

Abstract: Parikh Fuzzy vector for finite words of Rectangular Hilbert Space Filling Curve is introduced. Recurrence relations for this vector and its complement vector are produced. It is shown that the components of the Parikh Fuzzy vector are equally distributed at the limiting level. Hence, it is observed that the Parikh Fuzzy vector tends to a constant vector as n tends to infinity. Moreover, it is also valid to any other kinds of Space Filling Curves like Lebesgue Space Filling Curve, Peano Curve and Moore Curve

Keywords and Phrases: Finite word, Fuzzy vector, Rectangular Hilbert Space Filling Curve, Parikh vector.

1. Introduction

Space Filling Curves are applied to visit each cell of a multidimensional grid exactly once. These concepts of traversal are very useful in image processing, data organization and in reducing dimensions of multidimensional data. Generally Space Filling Curves fill a square using iterative process. For a particular case, Rectangular Space Filling curves fill a rectangle by using recursive progression.

In [1] the concept of fuzzy basis of fuzzy vector space is studied. The authors of [2] studied the nature of binary alphabets in Parikh matrix mapping. Combinations and selections on words are explained in [3]. The authors of [4] provided the notion of Parikh prime words. The concept of Parikh factor matrix is introduced in [5].

The properties of recurrence relations of Parikh vectors for finite words are discussed in [6] and [8]. The authors of [7] formed different representations of fuzzy vectors. Finite words for Space Filling Curves are investigated in [9] and [10].

Finite words for Rectangular Hilbert Space Filling Curves (RHSFC) are specified from [5] in the second section. Parikh fuzzy vectors for these words are defined in third section. Also properties of Parikh fuzzy vector for finite words of Rectangular Hilbert Space Filling Curve are discussed in the third section. Finally the limiting case of the Parikh fuzzy vector is analyzed.

2. RHSFC And Finite Words

The construction of the Rectangular Hilbert Space Filling Curve is observed from [5] and n^{th} finite iteration of this curve is described by the String W_n.

Let $W_1 = \bar{u}\, r\, \bar{d}$

This implies $W_2 = \bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\;\; u\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\;\; d\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}$

$|W_n| = 4(9)^{n-1} - 1$

$$|W_n|_{\bar{u}} = |W_n|_{\bar{d}} = \begin{cases} \dfrac{3^{n-1}(3^{n-1}-1)}{2}, & if \quad n \quad is \quad even \\ \dfrac{3^{n-1}(3^{n-1}+1)}{2}, & if \quad n \quad is \quad odd \end{cases}$$

$$|W_n|_{\bar{r}} = |W_n|_{\bar{\ell}} = |W_n|_u = |W_n|_d = \begin{cases} \dfrac{3^{n-1}(3^{n-1}+1)}{2}, & if \quad n \quad is \quad even \\ \dfrac{3^{n-1}(3^{n-1}-1)}{2}, & if \quad n \quad is \quad odd \end{cases}$$

$$|W_n|_r = \begin{cases} \dfrac{3^{n}(3^{n-2}+1)}{2} - 1, & if \quad n \quad is \quad odd \\ \dfrac{3^{n-1}(3^{n-1}+1)}{2} - 1, & if \quad n \quad is \quad even \end{cases}$$

$$|W_n|_{\ell} = \begin{cases} \dfrac{3^{n-1}(3^{n-1}-1)}{2}, & if \quad n \quad is \quad odd \\ \dfrac{3^{n}(3^{n-2}-1)}{2}, & if \quad n \quad is \quad even \end{cases}$$

3. Parikh Fuzzy Vector

Definition 3.1.

Parikh FuzzyVector:

Let Σ = { a_1< a_2 < ….<a_k} be an ordered alphabet. The Parikh fuzzy mapping is a mapping $p_f : \Sigma^* \to [0\,1]^k$ *defined as* $p_f(w) = \left(p_w(a_1), p_w(a_2), p_w(a_3)......p_w(a_k) \right)$ *where* $p_w(a_i)$ *is the probability of occurrences of* a_i *in w. i.e.* $p_w(a_i) = \dfrac{|w|_{a_i}}{|w|}$

Definition 3.2.

Complement Parikh Fuzzy Vector :

Let Σ = { a_1< a_2 < ….<a_k} be an ordered alphabet. The Complement Parikh fuzzy mapping is a mapping $c_f : \Sigma^* \to [0\,1]^k$ *defined as*

$$c_f(w) = \left(1 - p_w(a_1), 1 - p_w(a_2), ...1 - p_w(a_k) \right)$$

Example 1: *Let Σ ={a < b} be an ordered alphabet. Then*

$$p_f(abaa) = (0.75, 0.25), \quad c_f(abaa) = (0.25, 0.75)$$

Example 2 : *Let Σ ={a < b < c} be an ordered alphabet. Then*

$$p_f(abaa) = (0.75, 0.25, 0) \qquad c_f(abaa) = (0.25, 0.75, 1)$$

4. Parikh Fuzzy Vector Of Wn

Let the alphabet Σ of W_n is ordered by $\bar{u} < u < r < \bar{r} < \bar{d} < d < \ell < \bar{\ell}$.

Then the Parikh Fuzzy vector of W_n is given by

$$p_f(W_n) = \left(p_{W_n}(\bar{u}), p_{W_n}(u), p_{W_n}(r), p_{W_n}(\bar{r}), p_{W_n}(\bar{d}), p_{W_n}(d), p_{W_n}(\ell), p_{W_n}(\bar{\ell}) \right)$$

(4.1) $$p_f(W_n) = \left(\frac{|W_n|_{\bar{u}}}{|W_n|}, \frac{|W_n|_{u}}{|W_n|}, \frac{|W_n|_{r}}{|W_n|}, \frac{|W_n|_{\bar{r}}}{|W_n|}, \frac{|W_n|_{\bar{d}}}{|W_n|}, \frac{|W_n|_{d}}{|W_n|}, \frac{|W_n|_{\ell}}{|W_n|}, \frac{|W_n|_{\bar{\ell}}}{|W_n|} \right)$$

When n=1 in (4.1)

$$p_f(W_1) = \left(\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0, 0, 0 \right)$$

RECURRENCE RELATION FOR $p_f(W_n)$

Parikh Fuzzy vector $p_f(W_n)$ of W_n can be recursively written as

$$p_f(W_{n+1}) = \frac{9w_n}{w_{n+1}} p_f(W_n) + \frac{k(n)}{w_{n+1}}$$

$$\boxed{w_{n+1} p_f(W_{n+1}) = 9w_n p_f(W_n) + k(n)}$$

where

$$k(n)=\begin{cases}\left(2(-3)^n,-2(-3)^n,8-4(3)^n,-2(-3)^n,2(-3)^n,-2(-3)^n,0,-2(-3)^n\right), \\ \quad if\ n\ is\ odd \\ \left(2(-3)^n,-2(-3)^n,8,-2(-3)^n,2(-3)^n,-2(-3)^n,4(3)^n,-2(-3)^n\right), \\ \quad if\ n\ is\ even\end{cases}$$ and $w_n = |W_n|$

with initial condition

$$p_f(W_1)=\left(\frac{1}{3},0,\frac{1}{3},0,\frac{1}{3},0,0,0\right)\ when\ n=1\ in\ (4.1)$$

The recurrence equation is linear non-homogeneous non-autonomous equation with variable coefficients.

RECURRENCE RELATION FOR COMPLEMENT PARIKH FUZZY VECTOR $c_f(W_n)$ OF $p_f(W_n)$

The complement of $p_f(W_n)$ is given by

(4.2) $c_f(W_n) = \left(1-p_{H_n}(\bar{u}),1-p_{H_n}(u),1-p_{H_n}(r),1-p_{H_n}(\bar{r}),1-p_{H_n}(\bar{d}),1-p_{H_n}(d),1-p_{H_n}(\ell),1-p_{H_n}(\bar{\ell})\right)$

It is also a fuzzy vector since its values ∈ [0, 1]

Therefore $c_f(W_1)=\left(\frac{2}{3},1,\frac{2}{3},1,\frac{2}{3},1,1,1\right) when\ n=1\ in\ (4.2)$

The Complement Parikh Fuzzy vector $c_f(W_n)$ of W_n can be recursively written as

$$\bar{1}-p_f(W_{n+1})+\frac{9w_n}{w_{n+1}}\bar{1}=\bar{1}+\frac{9w_n}{w_{n+1}}\bar{1}-\frac{9w_n}{w_{n+1}}p_f(W_n)-\frac{k(n)}{w_{n+1}}\ c_f(W_{n+1})+\frac{9w_n}{w_{n+1}}\bar{1}=\bar{1}+\frac{9w_n}{w_{n+1}}c_f(W_n)-\frac{k(n)}{w_{n+1}}$$

$$w_{n+1}c_f(W_{n+1})=w_{n+1}\bar{1}-9w_n\bar{1}+9w_nc_f(W_n)-k(n)$$

$$\boxed{w_{n+1}c_f(W_{n+1})=9w_nc_f(W_n)+(w_{n+1}-9w_n)\bar{1}-k(n)}$$

$$w_{n+1}c_f(W_{n+1})=9w_nc_f(W_n)+8\bar{1}-k(n)$$

since $w_n= 4(9)^{n-1} – 1$

$w_{n+1}= 4(9)^n – 1=9×4(9)^{n-1} – 1= 9(w_n+1)-1$

$w_{n+1}= 9 w_n+8$ implies $w_{n+1}- 9 w_n=8$

$\bar{1}$ =(1,1,1,1,1,1,1,1) and $w_n = |W_n|$ with initial condition

$$c_f(W_1)=\left(\frac{2}{3},1,\frac{2}{3},1,\frac{2}{3},1,1,1\right) when\ n=1\ in\ (4.2)$$

The recurrence equation is linear non-homogeneous non-autonomous equation with variable coefficients.

UPPER BOUND OF $p_f(W_n)$

The largest element in the fuzzy vector 'a' is called its upper bound.

$$\widehat{a}=\max_{i\in[1,n]}[a_i]$$ where $a=(a_1,a_2,a_3,....a_n)$

Therefore $\widehat{p_f(W_n)} = \begin{cases}p_{W_n}(\bar{r}), & if\ n\ is\ even \\ p_{W_n}(r), & if\ n\ is\ odd\end{cases}$

LOWER BOUND OF $p_f(W_n)$

The smallest element in the fuzzy vector 'a' is called as its lower bound.

$$\underset{\frown}{a} = \min_{i \in [1,n]} [a_i] \quad \textbf{where} \quad a = (a_1, a_2, a_3, \ldots a_n)$$

Therefore $\underset{\smile}{p_f(W_n)} = p_{W_n}(\ell)$

5. Limiting Case Of Pf(Wn)

The values of p(a) where $a \in \sum = \{u, d, r, l, \bar{u}, \bar{d}, \bar{r}, \bar{l}\}$ **are listed in Table1.**

Table 1.Probabilites of occurrences of the letters

n	$p_{W_n}(r)$	$p_{W_n}(\ell)$	$p_{W_n}(\bar{u}) = p_{W_n}(\bar{d})$	$p_{W_n}(u) = p_{W_n}(\bar{r}) = p_{W_n}(d) = p_{W_n}(\bar{\ell})$
1	0.333333333	0	0.333333333	0
2	0.142857143	0	0.085714286	0
3	0.164086687	0.111455	0.139318885	0.111455
4	0.129331046	0.111149	0.120411664	0.111149
5	0.129596464	0.123461	0.126548032	0.123461
6	0.125510701	0.123457	0.124486124	0.123457
7	0.125513992	0.124829	0.125171527	0.124829
8	0.12505711	0.124829	0.124942851	0.124829
9	0.125057151	0.124981	0.125019053	0.124981
10	0.12500635	0.124981	0.124993649	0.124981
11	0.125006351	0.124998	0.125002117	0.124998
12	0.125000706	0.124998	0.124999294	0.124998
13	0.125000706	0.125	0.125000235	0.125
14	0.125000078	0.125	0.124999922	0.125
15	0.125000078	0.125	0.125000026	0.125
16	0.125000009	0.125	0.124999991	0.125
17	0.125000009	0.125	0.125000003	0.125
18	0.125000001	0.125	0.124999999	0.125
19	0.125000001	0.125	0.125	0.125
20	0.125	0.125	0.125	0.125
21	0.125	0.125	0.125	0.125
22	0.125	0.125	0.125	0.125
23	0.125	0.125	0.125	0.125
24	0.125	0.125	0.125	0.125
25	0.125	0.125	0.125	0.125

From this table, it can be seen that the probabilities of occurrences of the eight letters are approximately equal to 0.125 after some iterations. But, it can be noticed that letters $u, d, l\ \bar{r}\ and\ \bar{l}$ tend to their limiting value 0.125 faster than the other letters.. Therefore, the occurrences of letters of W_n are equally probably distributed as n tends to infinity. Moreover, it can be applicable to any formation of finite words for any Space Filling Curve. That is, if the finite words are formed with k letters, then the probability of occurrences of these letters are equal to 1/k at its limiting case. Hence the Parikh Fuzzy vector tends to a constant vector as n tends to infinity.

Theoretical View For Limiting Value Of Pf (Wn)

The limiting value of $p_f(W_n)$ can be found by applying limit $n \to \infty$ to $p_f(W_n)$. Firstly the limit value of $p_{W_n}(\bar{u})$ can be found as follows.

$$\lim_{n\to\infty} p_{W_n}(\bar{u}) = \lim_{n\to\infty} \frac{3^{2n-2} - 3^{n-1}}{2\times\left(4(9)^{n-1}-1\right)} = \lim_{n\to\infty} \frac{3^{2n-2}\left(1-\frac{1}{3^{n-1}}\right)}{2\times 4(3)^{2n-2}\left(1-\frac{1}{4(3)^{2n-2}}\right)} = \frac{1}{8} = 0.125$$

Similarly, other limit values of probabilities for other letters namely u,d,r, $l, \overline{d}, \overline{r}$ and $\overline{l}$. Therefore Parikh Fuzzy vector tends to (0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125) as $n \to \infty$.

6. Conclusion

Parikh Fuzzy vector of a word over an ordered alphabet with finite number of letters was introduced. Parikh Fuzzy vectors are computed correspondingly for finite words of Rectangular Hilbert Space Filling Curve. It is observed that, this vector tends to a constant vector as n tends to infinity. Additionally, this nature is also true for other kinds of Space Filling Curve. Also some of the properties of these vectors were analyzed.

7. Further Research

Some more properties of Fuzzy Parikh Vectors have to be discussed further.

Acknowledgment

The authors would like to thank Prof. Dr. Ponnammal Natarajan, Former Director of Research, Anna University, Chennai, India, for her intuitive ideas and fruitful discussions with respect to the paper's contribution..

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