Research Article

Tree Domination Number Of Middle And Splitting Graphs

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Abstract: Let G = (V, E) be a connected graph. A subset D of V is called a dominating set of G if N[D] = V. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set D of a graph G is called a tree dominating set (ntr - set) if the induced subgraph $\langle D\rangle$ is a tree. The tree domination number $\gamma_{tr}(G)$ of G is the minimum cardinality of a tree dominating set. The Middle Graph M(G) of G is defined as follows. The vertex set of M(G) is $V(G)\cup E(G)$. Two vertices x. y in the vertex set of M(G) are adjacent in M(G) if one of the following holds. (i) x, y are in E(G) and x, y are adjacent in G. (ii) $x\in V(G)$, $y\in E(G)$ and y is incident at x in G. Let G be a graph with vertex set V(G) and let V′(G) be a copy of V(G). The splitting graph S(G) of G is the graph, whose vertex set is $V(G) \cup V'(G)$ and edge set is {uv, u′v and uv′: $uv\in E(G)$}. In this paper we study the concept of tree domination in middle and splitting graphs.

Keywords: Domination number, connected domination number, tree domination number, middle graph, splitting graph.

Mathematics Subject Classification: 05C69

1 INTRODUCTION

The graphs considered here are nontrivial, finite and undirected. The order and size of G are denoted by n and m respectively. If $D \subseteq V$, then $N(D) = \bigcup_{v\in D} N(v)$ and $N[D] = N(D) \cup D$ where N(v) is the set of vertices of G which are adjacent to v. The concept of domination in graphs was introduced by Ore[4].

The graph G o K_1 is obtained from the graph G by attaching a pendent edge to all the vertices of G. The total graph T(G) of a graph G is a graph such that the vertex set T(G) corresponds to the vertices and edges of G and two vertices are adjacent in T(G) if and only if their corresponding elements are either adjacent or incident in G. A covering graph is a subgraph which contains either all the vertices or all the edges corresponding to some other graph. A subgraph which contains all the vertices is called a line(edge) covering. A subgraph which contains all the edges is called a vertex covering. The Middle Graph M(G) of G is defined as follows. The vertex set of M(G) is $V(G)\cup E(G)$. Two vertices x. y in the vertex set of M(G) are adjacent in M(G) if one of the following holds. (i) x, y are in E(G) and x, y are adjacent in G. (ii) $x\in V(G)$, $y\in E(G)$ and y is incident at x in G. Let G be a graph with vertex set V(G) and let V′(G) be a copy of V(G). The splitting graph S(G) of G is the graph, whose vertex set is $V(G) \cup V'(G)$ and edge set is {uv, u′v and uv′: $uv\in E(G)$}.

A subset D of V is called a dominating set of G if N[D] = V. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. Xuegang Chen, Liang Sun and Alice McRac [9] introduced the concept of tree domination in graphs. A dominating set D of G is called a tree dominating set, if the induced subgraph $\langle D\rangle$ is a tree. The minimum cardinality of a tree dominating set of G is called the tree domination number of G and is denoted by $\gamma_{tr}(G)$. In this paper we study the concept of tree domination in middle and splitting graphs.

2. PRIOR RESULTS

Theorem 2.1: [2] For any graph G, $\kappa(G) \le \delta(G)$.

Theorem 2.2: [9] For any connected graph G with $n \ge 3$, $\gamma_{tr}(G) \le n - 2$.

Theorem 2.3: [9] For any connected graph G with $\gamma_{tr}(G) = n - 2$ iff $G \cong P_n$ (or) C_n.

Theorem 2.4: [9] For every support is a member of every tree dominating set of G, $\gamma_{tr}(G) = s$, where S is the set of support vertices and $|S| = s$.

Theorem 2.5: [9] For every connected graph G with n vertices, $\gamma_{tr}(G) = n - 2$ if and only if G is isomorphic to P_n or C_n.

3. MAIN RESULTS

In this section, tree domination numbers of middle and splitting graphs are found.

3.1. TREE DOMINATION NUMBER IN MIDDLE GRAPHS

The Middle Graph M(G) of G is defined as follows. The vertex set of M(G) is $V(G) \cup E(G)$. Two vertices x. y in the vertex set of M(G) are adjacent in M(G) if one of the following holds.

(i) x, y are in E(G) and x, y are adjacent in G.

(ii) $x \in V(G)$, $y \in E(G)$ and y is incident at x in G.

In this section, tree domination numbers for middle graphs of some particular graphs are found and the graphs for which $\gamma_{tr}(M(G)) = 1, 2$ and $n - 2$ are characterized.

Example 3.1.1:

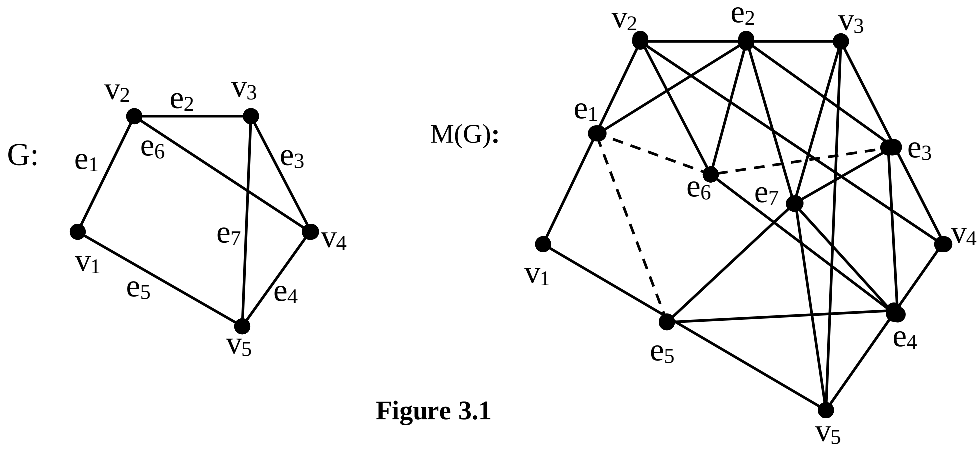


Figure 3.1

In the graph M(G) given in Figure 3.1, $\{e_1, e_3, e_5, e_6\}$ is a minimum tree dominating set and $\gamma_{tr}(M(G)) = 4$.

Example 3.1.2:

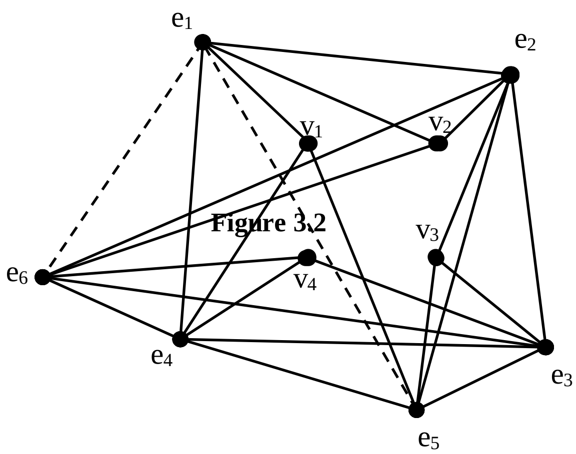


Figure 3.2

In the graph $M(K_4)$ given in Figure 3.2, a minimum tree dominating set is $\{e_1, e_5, e_6\}$ and $\gamma_{tr}(M(K_4)) = 3$.

Theorem 3.1.1:

For any path P_n on n vertices, $\gamma_{tr}(M(P_n)) = n - 1$, $n \geq 3$.

Proof:

The set $L(P_n)$ is a minimum tree dominating set of $M(P_n)$, since $\langle L(P_n) \rangle$ is isomorphic to P_{n-1} and each vertex of G in M(G) is adjacent to atleast one vertex in $L(P_n)$. Therefore, $\gamma_{tr}(M(P_n)) = |V(L(P_n))| = n - 1$, $n \geq 3$.

Theorem 3.1.2:

For any cycle C_n on n vertices, $\gamma_{tr}(M(C_n)) = n - 1$, $n \geq 3$.

Proof:

Let $e \in V(L(C_n))$. The set $L(C_n) - \{e\}$ is a minimum tree dominating set of $M(C_n)$ and $\gamma_{tr}(M(C_n)) = n - 1$, $n \geq 3$.

Theorem 3.1.3:

$\gamma_{tr}(M(K_{1,n})) = 0$, $n \geq 3$.

Proof:

The pendant vertices of $K_{1,n}$ are the pendant vertices of $M(K_{1,n})$. The vertices of $M(K_{1,n})$ adjacent to pendant vertices are vertices of $L(K_{1,n})$. But the subgraph of $M(K_{1,n})$ induced by vertices of L(G) is a complete graph. Since any tree dominating set of $M(K_{1,n})$ contains all supports, there exists no tree dominating set for $M(K_{1,n})$ and hence $\gamma_{tr}(M(K_{1,n})) = 0, n \geq 3$.

Theorem 3.1.4:

$\gamma_{tr}(M(P_n \circ K_1)) = 0, n \geq 2$, where $P_n \circ K_1$ is the Corona of P_n with K_1.

Proof:

The pendant vertices of $P_n \circ K_1$ are pendant vertices of $M(P_n \circ K_1)$. The supports are the vertices in $M(P_n \circ K_1)$ corresponding to pendant edges in $P_n \circ K_1$. Any dominating set of $M(P_n \circ K_1)$ contains all these supports. To get a tree dominating set of $M(P_n \circ K_1)$, vertices corresponding to edges of P_n in $P_n \circ K_1$ is to be included. But the subgraph of $M(P_n \circ K_1)$ induced by this dominating set contains cycles. Therefore, there exists no tree dominating set for $M(P_n \circ K_1)$ and hence $\gamma_{tr}(M(P_n \circ K_1)) = 0, n \geq 2$.

Theorem 3.1.5:

$\gamma_{tr}(M(\overline{P_n})) = n - 1$, where $\overline{P_n}$ is the complement of P_n, $n \geq 5$.

Proof:

Let $V(\overline{P_n}) = \{ v_1, v_2, v_3, \dots , v_n\}$ and let $e_{i,j} = (v_i, v_{i+j})$, $i = 1, 2, 3, \dots , n-2$ and $j = 2, 3, \dots , n-i$ and $e_{1,n} = (v_1, v_n)$ be the edges of $\overline{P_n}$.

Then $v_1, v_2, \dots ,v_n, e_{i,j} \in V(M(\overline{P_n}))$.

Case 1. n is even

Let $D = \{e_{1,(n+2)/2}, e_{1,(n+4)/2}, e_{2,(n+4)/2}, e_{2,(n+6)/2}, e_{3,(n+6)/2}, e_{3,(n+8)/2}, \dots, e_{(n-2)/2,\, n-1}, e_{(n-2)/2,\, n}, e_{n/2,\, n}\}$. Then $D \subseteq V(M(\overline{P_n}))$. D dominates the vertices of $L(\overline{P_n})$ in $M(\overline{P_n})$. The vertices $e_{1,(n+2)/2}$, $e_{1,(n+4)/2}$ dominate v_1, $v_{(n+2)/2}$ and $v_{(n+4)/2}$; $e_{2,(n+6)/2}$ dominates v_2 and $v_{(n+6)/2}$; $e_{3,(n+8)/2}$ dominates v_3 and $v_{(n+8)/2}$; ….; $e_{n/2,n}$ dominates $v_{n/2}$ and v_n. Therefore, D is a dominating set of $\overline{P_n}$. Also, $\langle D\rangle$ is a path on $n-1$ vertices and hence D is a tree dominating set of $M(\overline{P_n})$. Therefore, $\gamma_{tr}(M(\overline{P_n})) \leq |D| = n - 1$. Let D′ be a tree dominating set of $M(\overline{P_n})$. To dominate all the vertices of $M(\overline{P_n})$, D′ contains atleast (n/2) vertices and for $\langle D'\rangle$ is to be a tree, atleast $(n-2)/2$ vertices are to be added with D′. Therefore, D′ contains atleast $n-1$ vertices and $|D'| \geq n-1$ and hence $\gamma_{tr}(M(\overline{P_n})) = n-1$.

Case 2. n is odd.

The set $D = \{e_{1,(n+1)/2}, e_{1,(n+3)/2}, e_{2,(n+3)/2}, e_{2,(n+5)/2}, e_{3,(n+5)/2}, e_{3,(n+7)/2}, \dots, e_{(n-1)/2,\, n-1}, e_{(n-1)/2,\, n}\}$ is a dominating set of $M(\overline{P_n})$. Also, $\langle D\rangle$ is a path on $n-1$ vertices. As in Case 1, D is a minimum tree dominating set of $M(\overline{P_n})$ and hence $\gamma_{tr}(M(\overline{P_n})) = |D| = n-1$.

As in Theorem 2.2.5, the following can be proved.

Theorem 3.1.6.

$\gamma_{tr}(M(\overline{C_n})) = n - 1$, where $\overline{C_n}$ is the complement of C_n, $n \geq 5$.

In the following, the connected graphs G for which $\gamma_{tr}(M(G)) = 1, 2$ are characterized.

Theorem 3.1.7.

For any connected graph G, $\gamma_{tr}(M(G)) = 1$ if and only if $G \cong K_2$.

Proof:

When $G \cong K_2$, $\gamma_{tr}(M(G)) = 1$.

Assume $\gamma_{tr}(M(G)) = 1$. Let D be a tree dominating set of M(G) such that $|D| = 1$. If the vertex of D is a vertex of G, then $G \cong K_1$, since subgraph of M(G) induced by vertices of G is totally disconnected. If the vertex of D is a vertex of L(G), then $G \cong K_2$.

Theorem 3.1.8.

For any connected graph G on atleast three vertices, $\gamma_{tr}(M(G)) = 2$ if and only if there exists two adjacent edges e_1 and e_2 in G such that

(i) $\{e_1, e_2\}$ is an edge cover of G and

(ii) all the edges of G are adjacent to atleast one of e_1 and e_2.

Proof:

Assume $\gamma_{tr}(M(G)) = 2$. Let D be a tree dominating set of M(G) such that $|D| = 2$. Since the subgraph of M(G) induced by vertices of G is totally disconnected, either two vertices of D are vertices of L(G) (or) one vertex is in G and the other vertex is in L(G).

Case 1. Two vertices of D are vertices of L(G)

Let $e_1, e_2 \in D$. Then e_1, e_2 are edges in G. Let $e_3 \in E(G)$ be such that e_3 is not adjacent to both e_1 and e_2 in G. Then $e_3 \in L(G)$ is not adjacent to any of e_1 and e_2. Therefore, all the edges are adjacent to atleast one of e_1 and e_2.

Let u be a vertex of G in M(G). Then u is adjacent to one of e_1 and e_2 in M(G). Therefore, $\{e_1, e_2\}$ is an edge cover of G.

Case 2. One vertex is in G and the other is in L(G)

Let D = {u, e} be a tree dominating set of M(G), where $u \in V(G)$ and $e \in V(L(G))$. Then $e \in E(G)$ is incident with u. Let e = (u, v), where $v \in V(G)$. Let e_1 be an edge of G adjacent to e and $e_1 = (v, w)$, where $w \in V(G)$. Then $w \in V(M(G))$ is not adjacent to any of u and e. Let $e_2 = (w, x) \in E(G)$ be not adjacent to e (w, $x \in V(G)$). Then none of e_2, w, x in M(G) is adjacent to any of u and e. Therefore, $G \cong K_2$. But, $\gamma_{tr}(M(K_2)) = 1$.

By Case 1 and Case 2, $\gamma_{tr}(M(G)) = 2$.

Conversely, assume the conditions (i) and (ii). Since $\{e_1, e_2\}$ is an edge cover of G, $\{e_1, e_2\} \subseteq V(M(G))$ dominates all the vertices of G. By (ii), $\{e_1, e_2\}$ dominates all the vertices of L(G) in M(G). Also, $\langle\{e_1, e_2\}\rangle \cong K_2$, $\{e_1, e_2\}$ is a minimum tree dominating set of M(G) and $\gamma_{tr}(M(G)) = 2$.

Theorem 3.1.9:

Let G be a connected graph with n vertices and m edges. Then $\gamma_{tr}(M(G)) = n + m - 2$ if and only if G is isomorphic to K_2.

Proof:

By Theorem 2.5., "For every connected graph G with n vertices, $\gamma_{tr}(G) = n - 2$ if and only if G is isomorphic to P_n or C_n", $\gamma_{tr}(M(G)) = n + m - 2$ if and only if M(G) is isomorphic to P_{n+m} or C_{n+m}. If G contains two adjacent edges, then M(G) contains a triangle. If $G \cong 2K_2$, then $M(G) \cong 2P_3$. Therefore, G contains exactly one edge and M(G) is isomorphic to P_3. Also, there is no graph G for which M(G) is a cycle.

Theorem 3.1.10:

Let G be a connected graph on atleast three vertices. Then any tree dominating set D of L(G) is a tree dominating set of M(G) if and only if the set D′ of edges in G corresponding to vertices in D is

(i) an edge cover of G

(ii) each edge in G is adjacent to atleast one of the edges in D′.

Proof:

Let D be a tree dominating set of L(G) and let D′ be the set of all edges of G corresponding to vertices in D.

Assume conditions (i) and (ii). By (i), D dominates all the vertices of G in M(G). By (ii), D dominates all the vertices of L(G) in M(G). Since ⟨D⟩ is a tree in M(G), D is also a tree dominating set of M(G).

Conversely, if D′ is not an edge cover of G, then there exists a vertex u in G not incident with any of the edges in D′. Then the vertex u in M(G) is not adjacent to any of the vertices in D. Let e be an edge not adjacent to any of the edges in D′. Then the vertex e in M(G) is not adjacent to any of the vertices in D. Therefore, conditions (i) and (ii) hold.

Theorem 3.1.11:

Let G be a connected graph on atleast three vertices. Any tree dominating set of M(G) contains atmost two vertices of G.

Proof:

Let D be a tree dominating set of M(G) such that D contains atleast three vertices of G. Let v_1, v_2, v_3 be any three vertices of G in D. Since the subgraph of M(G) induced by $\{v_1, v_2, v_3\}$ is totally disconnected, D contains vertices of L(G) such that the corresponding edges in G are incident with v_1, v_2, v_3. Since ⟨D⟩ is a tree in M(G), adjacent vertices in ⟨D⟩ are not the vertices of G. Let $e_1 = (v_1, v_2)$ and $e_2 = (v_2, v_4)$, where $v_4 \in V(G)$. Then e_1 and e_2 in V(L(G)) are adjacent in M(G) and ⟨D⟩ contains a cycle and is not a tree. Therefore, D contains atmost two vertices of G.

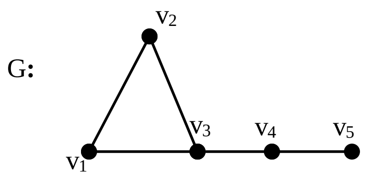
3.2. TREE DOMINATION NUMBER IN SPLITTING GRAPHS

In this section, tree domination numbers of splitting graphs of some standard graphs are obtained.

Definition 3.2.1:

Let G be a graph with vertex set V(G) and let V′(G) be a copy of V(G). The **splitting graph** S(G) of G is the graph, whose vertex set is $V(G) \cup V'(G)$ and edge set is $\{uv, u'v \text{ and } uv' : uv \in E(G)\}$.

Example 3.2.1:



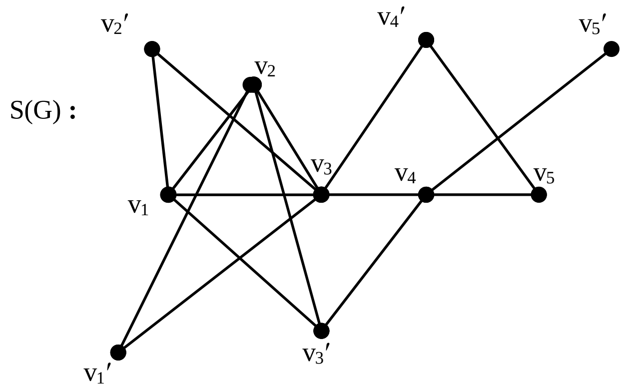


Figure 3.3

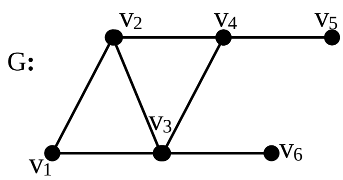
In the graph G given in Figure 2.4, the set {v_3, v_4} is a minimum tree dominating set of both G and S(G) and $\gamma(G) = \gamma_{tr}(G) = \gamma_{tr}(S(G)) = 2$.

Observation 3.2.1:

For any connected graph G, $\gamma_{tr}(G) \leq \gamma_{tr}(S(G))$.

This is illustrated by the following examples

Example 3.2.2:



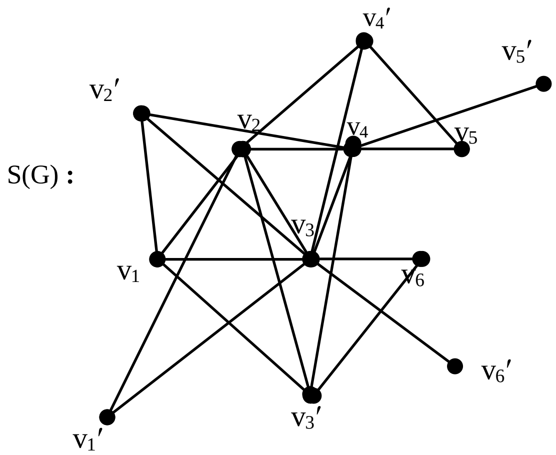
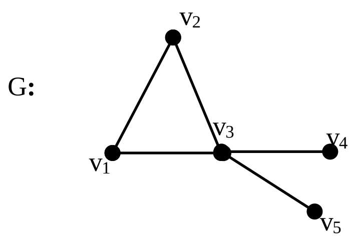


Figure 3.4

In the graph G given in Figure 3.4, the set {v_3, v_4} is a minimum tree dominating set of both G and S(G) and $\gamma_{tr}(G) = \gamma_{tr}(S(G)) = 2$.

Example 3.2.3:



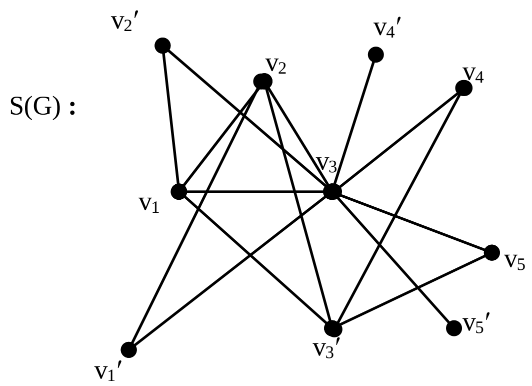


Figure 3.5

In the graph G given in Figure 2.7, minimum tree dominating set of G is {v_3} and $\gamma_{tr}(G) = 1$. In the graph S(G), minimum tree dominating set of S(G) is {v_1, v_3} and $\gamma_{tr}(S(G)) = 2$.

Therefore, $\gamma_{tr}(G) < \gamma_{tr}(S(G))$.

Theorem 3.2.1:

For the path P_n on n vertices, $\gamma_{tr}(S(P_n)) = n - 2$, $n \geq 4$.

Proof:

Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of P_n which are duplicated by the vertices $v_1', v_2', v_3', \ldots, v_n'$ respectively. The set $D = \{v_2, v_3, v_4, ..., v_{n-1}\}$ is a minimum dominating set of $S(P_n)$ and $\langle D \rangle \cong P_{n-2}$. Therefore, D is also a minimum tree dominating set of $S(P_n)$. Thus, $\gamma_{tr}(S(P_n)) = n - 2$.

Remark 3.2.1:

$\gamma_{tr}(S(P_2)) = 2$, $\gamma_{tr}(S(P_3)) = 2$.

Theorem 3.2.2: For the cycle C_n on n vertices, $\gamma_{tr}(S(C_n)) = n - 2$, $n \geq 4$.

Proof:

Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of C_n which are duplicated by the vertices $v_1', v_2', v_3', \ldots, v_n'$ respectively. The set $D = \{v_1, v_2, v_3, v_4, ..., v_{n-2}\}$ is a minimum dominating set of $S(C_n)$ and $\langle D \rangle \cong P_{n-2}$. Therefore, D is also a minimum tree dominating set of $S(C_n)$. Thus, $\gamma_{tr}(S(C_n)) = n - 2$.

Remark 3.2.2:

$\gamma_{tr}(S(C_3)) = 2$.

Theorem 3.2.3:

For the star $K_{1,n-1}$ on n vertices, $\gamma_{tr}(S(K_{1,n-1})) = 2$, $n \geq 2$.

Proof:

Let $v, v_1, v_2, v_3, \ldots, v_{n-1}$ be the vertices of star $K_{1,n-1}$ which are duplicated by the vertices $v', v_1', v_2', v_3', \ldots, v_{n-1}'$ respectively, where v is the central vertex of $K_{1,n-1}$. The set $D = \{v, v_1\}$ is a minimum dominating set of $S(K_{1,n-1})$ and $\langle D \rangle \cong K_2$. Therefore, D is a minimum tree dominating set of $S(K_{1,n-1})$.

Thus, $\gamma_{tr}(S(K_{1,n-1})) = 2$.

Theorem 3.2.4:

For the complete graph K_n on n vertices, $\gamma_{tr}(S(K_n)) = 2$, $n \geq 3$.

Proof:

Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of complete graph K_n which are duplicated by the vertices $v_1', v_2', v_3', \ldots, v_n'$ respectively. The set $D = \{v_1, v_2\}$ is a minimum dominating set of $S(K_n)$ and $\langle D \rangle \cong K_2$. Therefore, D is also a minimum tree dominating set of $S(K_n)$. Thus, $\gamma_{tr}(S(K_n)) = 2$.

Theorem 3.2.5:

For the complete bipartite graph $K_{r,s}$, $\gamma_{tr}(S(K_{r,s})) = 2$, $r, s \geq 2$.

Proof:

Let $A = \{v_1, v_2, v_3, \ldots, v_r\}$ and $B = \{u_1, u_2, u_3, \ldots, u_s\}$ be the set of vertices of bipartite graph $K_{r,s}$ which are duplicated by the vertices $v_1', v_2', v_3', .., v_r'$ and $u_1', u_2', u_3', \ldots, u_s'$ respectively. $D = \{v_1, u_1\}$ is a minimum dominating set of $S(K_{r,s})$ and $\langle D \rangle \cong K_2$. Therefore, D is also a minimum tree dominating set of $S(K_{r,s})$. Thus, $\gamma_{tr}(S(K_{r,s})) = 2$.

Theorem 3.2.6:

If $P_n \circ K_1$ is the Corona of P_n with K_1, then $\gamma_{tr}(S(P_n \circ K_1)) = n$, $n \geq 2$.

Proof:

Let $A = \{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices of P_n and $B = \{u_1, u_2, u_3, .., u_n\}$ be the set of pendant vertices adjacent to $v_1, v_2, v_3, ..., v_n$ respectively. Let $u_1', u_2', u_3', \ldots, u_n', v_1', v_2', v_3', \ldots, v_n'$ be the duplicated vertices of $u_1, u_2, u_3, .., u_n, v_1, v_2, \ldots, v_n$ respectively. $D = \{v_1, v_2, v_3, \ldots, v_n\}$ is a minimum dominating set of $S(P_n \circ K_1)$ and $\langle D \rangle \cong P_n$. Therefore, D is also a minimum tree dominating set of $S(P_n \circ K_1)$. Thus, $\gamma_{tr}(S(P_n \circ K_1)) = n$.

Theorem 3.2.7:

For the Wheel W_n on n vertices, $\gamma_{tr}(S(W_n)) = 2$, $n \geq 4$.

Proof:

Let $v, v_1, v_2, v_3, \ldots, v_{n-1}$ be the vertices of wheel W_n which are duplicated by the vertices $v_1', v_2', v_3', \ldots, v_n'$ respectively, where v is the central vertex of W_n and $v_1, v_2, v_3, v_4, \ldots, v_{n-1}$ be the vertices of C_{n-1}. $D = \{v, v_1\}$ is a minimum dominating set of $S(W_n)$ and $\langle D \rangle \cong K_2$. Therefore, D is a tree dominating set of $S(W_n)$. Thus, $\gamma_{tr}(S(W_n)) = 2$.

Theorem 3.2.8:

If $\overline{P_n}$ is the complement of P_n, then $\gamma_{tr}(S(\overline{P_n})) = 2$, $n \geq 2$.

Proof:

Let $v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices of $\overline{P_n}$. Let $v_1', v_2', v_3', \ldots, v_n'$ be the duplicated vertices of $v_1, v_2, v_3, \ldots, v_n$ respectively. The set $D = \{v_1, v_n\}$ is a minimum dominating set of $S(\overline{P_n})$ and $\langle D \rangle \cong K_2$. Therefore, D is also a tree dominating set of $S(\overline{P_n})$. Thus, $\gamma_{tr}(S(\overline{P_n})) = 2$.

Remark 3.2.3:

If $\gamma(G) = 1$, then $\gamma_{tr}(S(G)) = 2$. But the converse is not true. For example, for r, s $\geq$ 2, $\gamma_{tr}(S(K_{r,s})) = 2$, whereas $\gamma(K_{r,s}) \neq 1$.

Theorem 3.2.9.

Any tree dominating set of G containing atleast two vertices is also a tree dominating set of S(G).

Proof:

Let D be a tree dominating set of G. Then $\langle D\rangle$ is a tree and each vertex in V(G) – D is adjacent to atleast one vertex in D. Since $\langle D\rangle \subseteq V(S(G))$, $\langle D\rangle$ is also a tree in S(G). Each vertex of G in V(S(G)) – D is adjacent to atleast one vertex in D. Let $v \in V(G) – D$ and let v be adjacent to u in D. Then the duplicate vertex v′ of v is also adjacent to u. Since $|D| \geq 2$ and $\langle D\rangle$ is a tree, u is adjacent to atleast one vertex in $D \subseteq V(G)$. Let $w \in D$ be adjacent to u. Then the duplicate vertex u′ of u is adjacent to w and w′ is adjacent to u. Therefore, each vertex of V′(G) in V(S(G)) – D is adjacent to atleast one vertex in D of S(G) and D is also a tree dominating set of S(G).

Definition 3.2.2: Shadow Graph

Shadow Graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G′ and G″. Join each vertex u′ in G′ to the neighbours of the corresponding vertex u″ in G″.

Example 3.2.5:

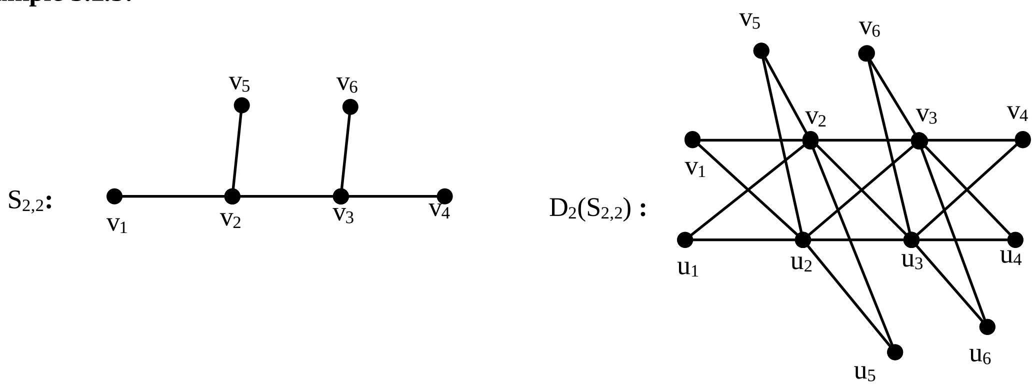


Figure 3.6

In the graph G and $D_2(G)$ given in Figure 3.6, the set $\{v_2, v_3\}$ is a minimum tree dominating set of both G and $D_2(G)$ and $\gamma_{tr}(G) = \gamma_{tr}(D_2(G)) = 2$.

Theorem 3.2.10:

Let G be a connected graph. Any tree dominating set of G containing atleast two vertices is also a tree dominating set of $D_2(G)$.

Proof:

Let D be a tree dominating set of G containing atleast two vertices and let G′ and G″ be two copies of G. Then D is a tree dominating set of G′. Let $u \in G′$ be such that $u \in D$ and $u″ \in G″$, Since D is a tree, u′ is adjacent to a vertex, say v in D. Then u″ is adjacent to v in D. Therefore, all the vertices in G″ is adjacent to atleast one vertex in D and hence D is a tree dominating set of $D_2(G)$.

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