Research Article

Error Estimates for a New H^1-Galerkin EMFEM of PIDEs with Nonlinear Memory

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Abstract: In this paper, the error estimate of H^1-Galerkin expanded mixed finite element methods (EMFEMs) is studied by investigating the semi discrete and fully discrete for parabolic integro-differential equations (PIDEs) with a nonlinear memory. We carried out theoretical survey for studying the existence and uniqueness of the numerical schemes.

Keywords: H^1- Galerkin method, mixed finite element method, semi and fully discrete, parabolic integro-differential equations, nonlinear memory.

I. INTRODUCTION

Consider the PIDE below with nonlinear memory [1]:

$$u_t - \Delta u + \int_0^t k(t-s)[-\nabla \cdot \big(a(x,u)\nabla u + b(x,u)\big) + c(x,u)\cdot \nabla u + g(x,u)]ds = f(x,t), (x,t) \in \Omega \times J,$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times j, \qquad (1.1)$$

$$u(x,0) = u_0(x), \qquad x \in \Omega,$$

Where, $\Omega \subset \mathbb{R}^d (d = 1,2,3)$ is a bounded domain in accompany of a smooth boundary $\partial\Omega$ and $J = (0,T]$ is interval of time with $0 < T < \infty$. The kernel k is positive definite and a smooth neither nonsmooth memory. f is a definite function. Consider the function $a(x,u)$ is a tensor one, $b(x,u)$ and $c(x,u)$ stand for vector ones and $g(x,u)$ represents the function of scalar one. Thus, the functions $a(x,u), b(x,u), c(x,u)$ and $g(x,u)$ stand as constant distinguishable for each variable too smooth as well as bounded. We take into consideration.

Equations of the class (1.1), or the linear types thereof, can appear in many material processes where it required to take in value the effects of memory due to the shortage of the frequent diffusion equations[2,3,4]. In order to find an approximate solution $\boldsymbol{u}$, much numerical methods are developed for solving such as these equations. finite element methods have been used widely for both linear or nonlinear integro-differential problems [5,6,7,8,9].

To the valuable mixed finite element methods (MFEMs). In [10], Sinha et al. investigated the semi discrete MFEMs for parabolic IDEs that seem via the modelling of nonlocal reactive streams in pored media as well as got a priori $\boldsymbol{L^2}$ error estimates for pressure and velocity can be happened in accompany of the two smooth and nonsmooth primary data. Ewing et al. [11] have derived maximum norm estimates and superconvergence results for mixed semi discrete adduction to PIDEs by mixed Ritz-Volterra projection and a way of close Green's function.

Regarding the H^1-Galerkin MFEM, Pani and Fairweather [12], H^1 -Galerkin MFEMs have studied the PIDEs that are used in mathematical tools. Error estimates can be got by semi discrete and discrete for equations with mono-dimension space. Shi et al [13], H^1 -Galerkin nonconforming MFEMs can be studied to PIDE. Through employing the standard property of the factors, we attain that the Galerkin mixed approximations obtain identical rates of convergence like the traditional mixed method, but not with LBB stability environment.

Likewise, H. Che et al. [14] H^1-Galerkin MFEM boned by extended mixed element method can be investigated by nonlinear pseudo-PIDEs. A priori error estimates are obtained for the unknown function, gradient function, and flux. They make theoretical analysis to discuss the existence and uniqueness of numerical solutions to the discrete scheme. H. Che et al. [15], H^1-Galerkin MFEM is discuss for nonlinear viscoelasticity equations based on H^1-Galerkin method and expanded mixed element method. The existence and uniqueness of solutions to the numerical scheme are proofed. A priori error estimation is obtained for the unknown function, the gradient function, and the flux. Furthermore, concerning a little MFEMs. Y. Liu et al. [16]. A new extended mixed procedure has been discussed and studied to linear PIDEs. The existence and uniqueness of solution for semi discrete method are proved as well as the fully discrete error estimates depend on backward Euler scheme can be classified. H. Li et al. [17], A new positive known EMFEM is advanced for PIDEs. Contrary to extended mixed method, the new extended mixed factor system is symmetric positive definite as well as the two-descent equation as well as the flux equation have been split out from its scalar indefinite equation. The presence and closeness for semi discrete have been got as well as error estimates have been confirmed to the two semi and fully discrete systems.

The aim of this study is for investigating the error estimates of a new H^1-Galerkin EMFEM of PIDE with nonlinear memory, it represents the approximation of four variables at once, since the scalar variable is approximated on the space $H^1(\Omega)$, and the other three vector variables are approximated on the space $H(div,\Omega)$. We action theoretical analysis to discuss the existence and uniqueness of numerical methods of the method and get error estimates for the fully discrete.

The outline of the research is as follows: In Section 2, a new H^1-Galerkin EMFEM of PIDE with nonlinear memory is present of weak formulation. In Section 3, we will error estimates for the fully discrete scheme of the H^1-Galerkin EMFEM are proved.

For brevity, through research we have used and will use the following expressions $a(u), b(u), c(u)$ and $g(u)$ instead of $a(x,u), b(x,u), c(x,u)$ and $g(x,u)$ respectively.

2. A New $\boldsymbol{H^1}$- Expanded mixed formulation

2.1 Mixed Weak Form

By rewritting the initial Governing problem as follows:

$$u_t - \nabla\cdot\left(\nabla u - \int_0^t k(t-s)\big(a(u)\nabla u + b(u)\big)d\tau\right) + \int_0^t k(t-s)\big(c(u)\cdot\nabla u + g(u)\big)d\tau = f(x,t), \qquad (2.1)$$

to explain the extended H^1-Galerkin MFEM . We classify PIDEs with nonlinear memory (2.1) to first-order system as below:

$$\boldsymbol{q} = \nabla u - \int_0^t k(t-s)\big(a(u)\nabla u + b(u)\big)d\tau, \quad \boldsymbol{p} = \int_0^t k(t-s)\big(c(u)\cdot\nabla u + g(u)\big)d\tau,$$

and

$$\boldsymbol{\sigma} = \nabla u,$$

Thus, (2.1) turns to

$$\begin{aligned}
&(a)\quad u_t - \nabla\cdot\boldsymbol{q} + \boldsymbol{p} = f\\
&(b)\quad \boldsymbol{\sigma} = \nabla u\\
&(c)\quad \boldsymbol{q} = \boldsymbol{\sigma} - \int_0^t k(t-s)\big(a(u)\boldsymbol{\sigma} + b(u)\big)d\tau \qquad (2.2)\\
&(d)\quad \boldsymbol{p} = \int_0^t k(t-s)\big(c(u)\cdot\boldsymbol{\sigma} + g(u)\big)d\tau,\\
&(e)\quad u(x,0) = u_0(x).
\end{aligned}$$

Let $W = H(div,\Omega) = \left\{\boldsymbol{w}\in\left(L^2(\Omega)\right)^d : \nabla\cdot\boldsymbol{w}\in L^2(\Omega)\right\}$, in addition to norm $\|\boldsymbol{w}\|_{H(div,\Omega)} = (\|\boldsymbol{w}\|^2 + \|\nabla\cdot\boldsymbol{w}\|^2)^{\frac{1}{2}}$ and $V = H_0^1(\Omega) = \{v\in H^1(\Omega): v = 0 \ \ on \ \ \partial\Omega\}$. First, multiply (3.2)(a) by $\nabla\cdot\boldsymbol{w}$ for $\boldsymbol{w}\in W$, and integrating on Ω, then apply the relation $(\psi,\nabla v) = -(\nabla\cdot\psi, v)$ (Divergence theorem) to the first term on the left side and substituting of derivative the equation (3.2)(b) to the final equation to have a weak from for (3.2)(a). then, multiplying (3.2)(b) by ∇v for $v\in H_0^1(\Omega)$, (3.2)(c) by $\boldsymbol{z}\in H(div,\Omega)$ as well as (3.2)(d) by $\boldsymbol{r}\in H(div,\Omega)$, then incorporating the resulting equations on Ω give to the variational formulation for (3.2)(b), (3.2)(c) and (3.2)(d). Thus, the variational formulation for (3.2) is to find $(u,\boldsymbol{\sigma},\boldsymbol{q},\boldsymbol{p}) \in H_0^1(\Omega)\times W\times W\times W$ such that

$$\begin{aligned}
&(a)\quad (\boldsymbol{\sigma}_t,\boldsymbol{w}) + (\nabla\cdot\boldsymbol{q},\nabla\cdot\boldsymbol{w}) - (\boldsymbol{p},\nabla\cdot\boldsymbol{w}) = -(f,\nabla\cdot\boldsymbol{w}), \qquad \forall\boldsymbol{w}\in W,\\
&(b)\quad (\boldsymbol{\sigma},\nabla v) = (\nabla u,\nabla v), \qquad \forall v\in H_0^1(\Omega),\\
&(c)\quad (\boldsymbol{q},\boldsymbol{z}) = (\boldsymbol{\sigma},\boldsymbol{z}) - \left(\int_0^t k(t-s)(a(u)\boldsymbol{\sigma},\boldsymbol{z})d\tau\right) + \left(\int_0^t k(t-s)(b(u),\boldsymbol{z})d\tau\right), \quad \forall\boldsymbol{z}\in W, \qquad (2.3)
\end{aligned}$$

$$(d)\quad (\boldsymbol{p},\boldsymbol{r}) = \left(\int_0^t k(t-s)(c(u)\cdot\boldsymbol{\sigma},\boldsymbol{r})d\tau\right) + \left(\int_0^t k(t-s)(g(u),\boldsymbol{r})d\tau\right), \qquad \forall \boldsymbol{r}\in W.$$

$$(e)\quad \boldsymbol{\sigma}(0) = \nabla u_0(x).$$

2. 2 Semi Discrete Scheme

The semi discrete MFEM for (2.3) is to determine find $\{u_h,\boldsymbol{\sigma}_h,\boldsymbol{q}_h,\boldsymbol{p}_h\}: [0,T] \mapsto V_h\times \boldsymbol{W}_h\times\boldsymbol{W}_h\times\boldsymbol{W}_h$ such that

$$(a)\ (\boldsymbol{\sigma}_{ht},\boldsymbol{w}_h) + (\nabla\cdot\boldsymbol{q}_h,\nabla\cdot\boldsymbol{w}_h) - (\boldsymbol{p}_h,\nabla\cdot\boldsymbol{w}_h) = -(f,\nabla\cdot\boldsymbol{w}_h), \qquad \forall \boldsymbol{w}_h\in\boldsymbol{W}_h,$$

$$(b)\ (\boldsymbol{\sigma_h},\nabla v_h) = (\nabla u_h,\nabla v_h), \qquad \forall v_h\in V_h,$$

$$(c)\ (\boldsymbol{q_h},\boldsymbol{z_h}) = (\boldsymbol{\sigma_h},\boldsymbol{z_h}) - \left(\int_0^t k(t-s)(a(u)\boldsymbol{\sigma_h},\boldsymbol{z_h})d\tau\right) + \left(\int_0^t k(t-s)(b(u_h),\boldsymbol{z_h})d\tau\right), \qquad \forall \boldsymbol{z_h}\in\boldsymbol{W}_h, \quad (2.4)$$

$$(d)\ (\boldsymbol{p_h},\boldsymbol{r_h}) = \left(\int_0^t k(t-s)(c(u)\cdot\boldsymbol{\sigma_h},\boldsymbol{r_h})d\tau\right) + \left(\int_0^t k(t-s)(g(u_h),\boldsymbol{r_h})d\tau\right), \qquad \forall \boldsymbol{r_h}\in\boldsymbol{W}_h.$$

$$(e)\quad \boldsymbol{\sigma}_h(0) = R_h\nabla u_0(x).$$

Where V_h and $\boldsymbol{W}_h$ stand as finite dimensional subspaces from V and $\boldsymbol{W}$**,** consequently**.** Thus,

$$V_h = \{v_h\in C^0(\Omega)\cap H_0^1 | v_h\in P_0(K), \forall K\in T_h\},$$
$$\boldsymbol{W}_h = \left\{\boldsymbol{w}_h\in \left(L^2(\Omega)\right)^2 | \boldsymbol{w}_h\in P_1(K), \forall K\in T_h\right\},$$

and Let T_h be a quasi-uniform family of partitioning of domain Ω. Let h_K refers to the diameter of K. Set $h = \max_{K\in T_h} h_K$.

2.3 Existence and Uniqueness

We will study the existence and uniqueness of solution for semi discrete scheme (2.4).

Theorem 2.1. There is a only discrete solution to the scheme (2.4)

Proof. Let $\{\varphi_i\}_{j=1}^N$ and $\{\psi_i(x)\}_{i=1}^M$ be bases of V_h and $\boldsymbol{W}_h$ respectively. Let

$$u_h = \sum_{i=1}^N u_i(t)\varphi_i(\boldsymbol{x}), \boldsymbol{\sigma}_h = \sum_{j=1}^M \sigma_j(t)\psi_j(\boldsymbol{x}),\ \boldsymbol{q}_h = \sum_{j=1}^M q_j(t)\psi_j(\boldsymbol{x}), \boldsymbol{p}_h = \sum_{j=1}^M p_j(t)\psi_j(\boldsymbol{x}), \quad (2.5)$$

and placing these terms to (2.4) as well as choosing $v_h = \varphi_k, k = 1,2,\dots,N,$, $\boldsymbol{w}_h = \boldsymbol{z}_h = \boldsymbol{r}_h = \psi_l, l = 1,2,\dots,M,$ thus (2.4) turns as:

$$(a)\qquad A\Sigma'(t) + BQ(t) - CP(t) = -F(t),$$
$$(b)\qquad D\Sigma(t) - EU(t) = 0,$$
$$(c)\qquad AQ(t) + A\Sigma(t) - \int_0^t k(t-s)M(U)\Sigma(t)d\tau - \int_0^t k(t-s)Rd\tau = 0, \quad (2.6)$$
$$(d)\qquad AP(t) - \int_0^t k(t-s)N(U)\Sigma(t)d\tau - \int_0^t k(t-s)Hd\tau = 0,$$

Where

$$A = (\psi_j,\psi_l)_{M\times M}, \qquad B = (\nabla\cdot\psi_j,\nabla\cdot\psi_l)_{M\times M},$$

$C = (\psi_j, \nabla \cdot \psi_l)_{M\times M}$, $F = (\mathrm{f}, \nabla \cdot \psi_l)_{1\times M}$,
$D = (\psi_j, \nabla\varphi_k)_{M\times N}$, $E = (\nabla\varphi_i, \nabla\varphi_k)_{N\times N}$,
$M(U) = (a(U)\psi_j, \nabla\psi_l)_{M\times M}$, $R = (\mathrm{b}(U), \psi_l)_{1\times M}$,
$M(U) = (c(U) \cdot \psi_j, \nabla\psi_l)_{M\times M}$, $H = (\mathrm{g}(U), \psi_l)_{1\times M}$,
$\Sigma = (\sigma_1, \sigma_2, \dots, \sigma_M)^T$, $Q = (q_1, q_2, \dots, q_M)^T$,
$P = (p_1, p_2, \dots, p_M)^T$, $U = (u_1, u_2, \dots, u_N)^T$,

The first value problems (2.6) can be written as follows:

$$
\begin{aligned}
&(a) \quad \Sigma'(t) + A^{-1}BQ(t) - A^{-1}CP(t) = -A^{-1}F(t),\\
&(b) \quad U(t) = E^{-1}D\Sigma(t),\\
&(c) \quad Q(t) + \Sigma(t) - A^{-1}\int_0^t k(t-s)M(U)\Sigma(t)d\tau - A^{-1}\int_0^t k(t-s)Rd\tau = 0, \qquad (2.7)\\
&(d) \quad P(t) - A^{-1}\int_0^t k(t-s)N(U)\Sigma(t)d\tau - A^{-1}\int_0^t k(t-s)Hd\tau = 0,
\end{aligned}
$$

replacing (2.7c) and (2.7d) into (2.7a) to have

$$
\Sigma'(t) = A^{-1}C\left(A^{-1}\int_0^t k(t-s)N(U)\Sigma(t)d\tau + A^{-1}\int_0^t k(t-s)Hd\tau\right)
$$
$$
-A^{-1}B\left(A^{-1}\int_0^t k(t-s)M(U)\Sigma(t)d\tau + A^{-1}\int_0^t k(t-s)Rd\tau\right) - A^{-1}F(t) + \Sigma(t) \qquad (2.8)
$$

Thus, by the differential equations theorem [18], (2.8) has a unique solution $\Sigma(t)$, then (2.7b), (2.7c) and (2.7d) has a unique solution $U(t), Q(t)$ and $P(t)$, respectively. Equivalently (2.3) has a unique solution.

3. New $\boldsymbol{H^1}$- Expanded mixed projection

For to discuss the convergence of the method, in start, we insert the new H^1- expanded mixed elliptic projection connected with our equations.

Let $(u_h, \boldsymbol{\sigma}_h, \boldsymbol{q}_h, \boldsymbol{p}_h): [0, T] \mapsto V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h$ be given as follows:

$$
\begin{aligned}
&(a) \quad (\nabla \cdot (q - q_h), \nabla \cdot \boldsymbol{w}_h) + (p_h - p, \nabla \cdot \boldsymbol{w}_h) = 0, && \forall \boldsymbol{w}_h \in \boldsymbol{W}_h,\\
&(b) \quad (\sigma - \sigma_h, \nabla v_h) - (\nabla(u - u_h), \nabla v_h) = 0, && \forall v_h \in V_h,\\
&(c) \quad (q - q_h, \boldsymbol{z}_h) - (\sigma - \sigma_h, \boldsymbol{z}_h) = 0, && \forall \boldsymbol{z}_h \in \boldsymbol{W}_h, \qquad (3.1)\\
&(d) \quad (p - p_h, \boldsymbol{r}_h) = 0, && \forall \boldsymbol{r}_h \in \boldsymbol{W}_h.
\end{aligned}
$$

Now, we will present some important lemmas.

4. Some Theorems

Theorem 4.1. There is the operator $R_h: H(div, \Omega) \longrightarrow \boldsymbol{W}_h$ like

$$(\boldsymbol{\sigma} - R_h\boldsymbol{\sigma}_h, \nabla v_h) = o, \qquad \forall v_h \in \boldsymbol{W}_h, \qquad (4.1)$$

and

$$\|\boldsymbol{\sigma} - R_h\boldsymbol{\sigma}_h\| \le ch^{k+1}\|\boldsymbol{\sigma}\|_{k+1}, \qquad (4.2)$$

Theorem 4.2. There is the operator $R_h: H(div, \Omega) \longrightarrow \boldsymbol{W}_h$ like

$$(\nabla \cdot (\boldsymbol{q} - \boldsymbol{q}_h), \nabla \cdot \boldsymbol{w}_h) = 0, \qquad \forall \boldsymbol{w}_h \in \boldsymbol{W}_h, \qquad (4.3)$$

and

$$\|\boldsymbol{q} - R_h\boldsymbol{q}_h\| \le ch^{k+1}\|\boldsymbol{q}\|_{k+1}, \qquad (4.4)$$

$$\|\nabla \cdot (\boldsymbol{q} - \boldsymbol{q}_h)\| \le ch^k\|\boldsymbol{q}\|_{k+1}, \qquad (4..5)$$

It has the evidence of the aforementioned theorems from [19,20].

Theorem 4.3. There is the operator $\Pi_h: H_0^1(\Omega) \longrightarrow V_h$ such that

$$(\nabla(u - u_h), \nabla v_h) = 0, \qquad \forall v_h \in V_h, \qquad (4.6)$$

and

$$\|u-\Pi_h u_h\| + h\|\nabla(u-u_h)\| \le ch^{m+1}\|u\|_{m+1}, \tag{4.7}$$

we can get the proof of the above theorem from [21].

5. Error Estimates The Semi-discrete Method

In this section, the study clarifies the convergence results and error estimates for the new H^1-Galerkin EMFEM given in this paper . To explain a priori error estimates, we get the errors as below:

$$\sigma-\sigma_h = \sigma - \mathrm{R}_h\sigma + \mathrm{R}_h\sigma - \sigma_h = \delta+\theta,$$
$$q-q_h = q - \mathrm{R}_h q + \mathrm{R}_h q - q_h = \alpha+\beta,$$
$$p-p_h = p - \mathrm{R}_h p + \mathrm{R}_h p - p_h = \rho+\xi,$$

and

$$u-u_h = u - \Pi_h u + \Pi_h u - u_h = \eta+\zeta.$$

to get the error equations, applying (2.3) and (2.4) with the projections (3.1)

$$(a)\ (\theta_t, \boldsymbol{w}_h) + (\nabla\cdot\beta, \nabla\cdot\boldsymbol{w}_h) - (\xi, \nabla.\boldsymbol{w}_h) = (\delta_t, \boldsymbol{w}_h), \qquad \forall \boldsymbol{w}_h \in \boldsymbol{W}_h,$$
$$(b)\ (\theta, \nabla v_h) - (\nabla\zeta, \nabla v_h) = 0, \qquad \forall v_h \in V_h,$$
$$(c)\ (\beta, \boldsymbol{z}_h) = (\theta, \boldsymbol{z}_h) - \left(\int_0^t k(t-s)\big(a(u)-a(u_h)\big)\boldsymbol{\sigma} d\tau, \boldsymbol{z}_h\right) - \left(\int_0^t k(t-s)a(u)\delta d\tau, \boldsymbol{z}_h\right)$$
$$- \left(\int_0^t k(t-s)a(u)\theta d\tau, \boldsymbol{z}_h\right) + \left(\int_0^t k(t-s)\big(b(u)-b(u_h)\big) d\tau, \boldsymbol{z}_h\right), \quad \forall \boldsymbol{z}_h \in \boldsymbol{W}_h, \tag{5.1}$$
$$(d)\ (\xi, \boldsymbol{r}_h) = \left(\int_0^t k(t-s)\big(c(u)-c(u_h)\big)\cdot\boldsymbol{\sigma} d\tau, \boldsymbol{r}_h\right) + \left(\int_0^t k(t-s)c(u)\cdot\delta d\tau, \boldsymbol{r}_h\right)$$
$$+ \left(\int_0^t k(t-s)c(u)\cdot\theta d\tau, \boldsymbol{r}_h\right) + \left(\int_0^t k(t-s)\big(g(u)-g(u_h)\big) d\tau, \boldsymbol{r}_h\right). \quad \forall \boldsymbol{r}_h \in \boldsymbol{W}_h,$$

note that the equation

$$\begin{aligned} a(u)\boldsymbol{\sigma} - a(u_h)\boldsymbol{\sigma}_h &= a(u)\boldsymbol{\sigma} - a(u_h)\boldsymbol{\sigma} + a(u_h)\boldsymbol{\sigma} - a(u_h)\boldsymbol{\sigma}_h \\ &= \big(a(u)-a(u_h)\big)\boldsymbol{\sigma} + a(u)(\sigma - \Pi_h\sigma + \Pi_h\sigma - \sigma_h) \\ &= \big(a(u)-a(u_h)\big)\boldsymbol{\sigma} + a(u)(\delta+\theta), \end{aligned}$$

similarly, we get

$$\begin{aligned} \big(c(u)-c(u_h)\big)\cdot\boldsymbol{\sigma} = c(u)\cdot\boldsymbol{\sigma} - c(u_h)\cdot\boldsymbol{\sigma} &= c(u)\cdot\boldsymbol{\sigma} - c(u_h)\cdot\boldsymbol{\sigma} + c(u_h)\cdot\boldsymbol{\sigma} - c(u_h)\cdot\boldsymbol{\sigma}_h \\ &= \big(c(u)-c(u_h)\big)\cdot\boldsymbol{\sigma} + c(u)\cdot(\sigma - \Pi_h\sigma + \Pi_h\sigma - \sigma_h) \\ &= \big(c(u)-c(u_h)\big)\cdot\boldsymbol{\sigma} + c(u)\cdot(\delta+\theta). \end{aligned}$$

Now, we derive the error estimates for semi discrete method.

Theorem 5.1 suppose that $\boldsymbol{\sigma}_h(0) = R_h\nabla u_0(x)$ and let $(u, \boldsymbol{\sigma}, \boldsymbol{q}, \boldsymbol{p})$ and $(u_h, \boldsymbol{\sigma}_h, \boldsymbol{q}_h, \boldsymbol{p}_h)$ are the solution of (2.3) and (2.4), respectively, then we have the following estimates

(a) $\|u-u_h\|_1 \le Ch^{\min(k+1,m)}$

(b) $\|\boldsymbol{p}-\boldsymbol{p}_h\| \le Ch^{\min(k+1,m+1)}$

(c) $\|\nabla\cdot(\boldsymbol{q}-\boldsymbol{q}_h)\| \le Ch^{\min(k,m+1)}$

(d) $\|u-u_h\| + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\| + \|\boldsymbol{q}-\boldsymbol{q}_h\| + \|\boldsymbol{p}-\boldsymbol{p}_h\| \le Ch^{\min(k+1,m+1)}$.

Proof. Let $u-u_h = u-\Pi_h u + \Pi_h u - u_h = \eta+\zeta$, $p-p_h = p - \mathrm{R}_h p + \mathrm{R}_h p - p_h = \rho+\xi$, $q-q_h = q - \mathrm{R}_h q + \mathrm{R}_h q - q_h = \alpha+\beta$, $\sigma-\sigma_h = \sigma - \mathrm{R}_h\sigma + \mathrm{R}_h\sigma - \sigma_h = \delta+\theta$,
an estimates of η, ρ, α and δ can be get it from (4.2), (4,4), (4.5) and (4.7), and now we find an estimates of ς, ξ, β and θ. Setting $\boldsymbol{z}_h = \beta$ in (5.1(c)) and $\boldsymbol{r}_h = \xi$ in (5.1(d)) then we add the two resulting equations

$$\|\beta\|^2 = (\theta, \beta) - \left(\int_0^t k(t-s)\big(a(u)-a(u_h)\big)\boldsymbol{\sigma} d\tau, \beta\right) - \left(\int_0^t k(t-s)a(u)\delta d\tau, \beta\right)$$

$$-\left(\int_0^t k(t-s)a(u)\theta d\tau,\beta\right)+\left(\int_0^t k(t-s)\left(b(u)-b(u_h)\right)d\tau,\beta\right). \tag{5.2}$$

$$\|\xi\|^2=\left(\int_0^t k(t-s)\big(c(u)-c(u_h)\big)\cdot\boldsymbol{\sigma}d\tau,\xi\right)+\left(\int_0^t k(t-s)c(u)\cdot\delta d\tau,\xi\right)$$
$$+\left(\int_0^t k(t-s)c(u)\cdot\theta d\tau,\xi\right)+\left(\int_0^t k(t-s)\left(g(u)-g(u_h)\right)d\tau,\xi\right). \tag{5.3}$$

$$\|\beta\|^2+\|\xi\|^2=(\theta,\beta)-\left(\int_0^t k(t-s)\big(a(u)-a(u_h)\big)\boldsymbol{\sigma}d\tau,\beta\right)-\left(\int_0^t k(t-s)a(u)\delta d\tau,\beta\right)$$
$$-\left(\int_0^t k(t-s)a(u)\theta d\tau,\beta\right)+\left(\int_0^t k(t-s)\left(b(u)-b(u_h)\right)d\tau,\beta\right)$$
$$+\left(\int_0^t k(t-s)\big(c(u)-c(u_h)\big)\cdot\boldsymbol{\sigma}d\tau,\xi\right)+\left(\int_0^t k(t-s)c(u)\cdot\delta d\tau,\xi\right)$$
$$+\left(\int_0^t k(t-s)c(u)\cdot\theta d\tau,\xi\right)+\left(\int_0^t k(t-s)\left(g(u)-g(u_h)\right)d\tau,\xi\right). \tag{5.4}$$

Using Young's inequalities with appropriately small ε and Cauchy-Schwartz inequalities to obtain

$$|(\theta,\beta)|\le c\|\theta\|^2+\varepsilon\|\beta\|^2, \tag{5.5}$$

$$\left|-\left(\int_0^t k(t-s)\big(a(u)-a(u_h)\big)\boldsymbol{\sigma}d\tau,\beta\right)\right|\le cc_1c_2\int_0^t(\|\eta\|^2+\|\zeta\|^2)d\tau+\varepsilon\|\beta\|^2, \tag{5.6}$$

where, c_1 depends on $k(t-s)$, c_2 depends on $\|\boldsymbol{\sigma}\|_{W^1_\infty(L^\infty)}$

$$\left|-\left(\int_0^t k(t-s)a(u)\delta d\tau,\beta\right)\right|\le cc_1c_3\int_0^t\|\delta\|^2\,d\tau+\varepsilon\|\beta\|^2, \tag{5.7}$$

c_3 depends on the bound of $a(u)$,

$$\left|-\left(\int_0^t k(t-s)a(u)\theta d\tau,\beta\right)\right|\le cc_1c_3\int_0^t\|\theta\|^2\,d\tau+\varepsilon\|\beta\|^2, \tag{5.8}$$

$$\left|\left(\int_0^t k(t-s)\left(b(u)-b(u_h)\right)d\tau,\beta\right)\right|\le cc_1\int_0^t(\|\eta\|^2+\|\zeta\|^2)d\tau+\varepsilon\|\beta\|^2, \tag{5.9}$$

$$\left|\left(\int_0^t k(t-s)\big(c(u)-c(u_h)\big)\cdot\boldsymbol{\sigma}d\tau,\xi\right)\right|\le c_1c_2\int_0^t(\|\eta\|^2+\|\zeta\|^2)d\tau+c_1c_2\int_0^t\|\xi\|^2d\tau, \tag{5.10}$$

$$\left|\left(\int_0^t k(t-s)c(u)\cdot\delta d\tau,\xi\right)\right|\le c_1c_4\int_0^t\|\delta\|^2\,d\tau+c_1c_4\int_0^t\|\xi\|^2d\tau, \tag{5.11}$$

Where, c_4 depends on the bound of $c(u)$.

$$\left|\left(\int_0^t k(t-s)c(u)\cdot\theta d\tau,\xi\right)\right|\le c_1c_4\int_0^t\|\theta\|^2\,d\tau+c_1c_4\int_0^t\|\xi\|^2d\tau, \tag{5.12}$$

$$\left|\left(\int_0^t k(t-s)\left(g(u)-g(u_h)\right)d\tau,\xi\right)\right| \le c_1\int_0^t (\|\eta\|^2+\|\zeta\|^2)d\tau + c_1\int_0^t \|\xi\|^2 d\tau. \tag{5.13}$$

replacing (5.5)-(5.13) into (5.4) we have

$$\|\beta\|^2+\|\xi\|^2 \le C_1\|\theta\|^2 + C_2\int_0^t (\|\eta\|^2+\|\zeta\|^2+\|\delta\|^2+\|\theta\|^2+\|\xi\|^2) \tag{5.14}$$

where $C_1 = C_1(c,\varepsilon)$ and $C_2 = C_2(c,c_1,c_2,c_3,c_4)$.
Now, we estimates of ζ and θ, selecting $v_h = \zeta$ in (5.1(b))

$$(\nabla\zeta,\nabla\zeta) = (\theta,\nabla\zeta),$$

applying the Young's inequality, to get

$$c\,\|\nabla\zeta\|^2 + \varepsilon\,\|\nabla\zeta\|^2 \le c\,\|\theta\|^2 + \varepsilon\,\|\nabla\zeta\|^2,$$

Thus,

$$\|\nabla\zeta\|^2 \le \|\theta\|^2. \tag{5.15}$$

Since $\zeta \in V_h \subset H_0^1(\Omega)$, after that, $\|\zeta\| \le c_0\|\nabla\zeta\|$, thus we have

$$\|\zeta\|^2 \le c_0\|\theta\|^2. \tag{5.16}$$

Thus, placing $\boldsymbol{w}_h = \theta$, in (5.1(a)) leads to

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + (\nabla\cdot\beta,\nabla\cdot\theta) - (\xi,\nabla.\theta) = (\delta_t,\theta),$$

Then, using ε-Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 \le c(\|\xi\|^2+\|\delta_t\|^2+\|\nabla\cdot\beta\|^2) + \varepsilon\|\nabla\cdot\theta\|^2 + \varepsilon\|\theta\|^2, \tag{5.17}$$

and efficiently small ε, must

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 \le c(\|\xi\|^2+\|\delta_t\|^2+\|\nabla\cdot\beta\|^2) + \|\nabla\cdot\theta\|^2 + \|\theta\|^2. \tag{5.18}$$

Integrating every expression of (5.5) regarding 't' from 0 to t, and $\theta(0) = 0$, we get

$$\|\theta\|^2 \le c\int_0^t (\|\xi\|^2+\|\delta_t\|^2+\|\nabla\cdot\beta\|^2+\|\nabla\cdot\theta\|^2+\|\theta\|^2)d\tau \tag{5.19}$$

By Gronwell's lemma, yields

$$\|\theta\|^2 \le c\int_0^t (\|\xi\|^2+\|\delta_t\|^2+\|\nabla\cdot\beta\|^2)d\tau \tag{5.20}$$

here, we need to estimate of $\nabla\cdot\beta$.
Considering $\boldsymbol{w}_h = \beta$ in (5.1(a)) we have

$$\|\nabla\cdot\beta\|^2 + (\theta_t,\beta) = (\delta_t,\beta) + (\xi,\nabla.\beta), \tag{5.21}$$

The study has Cauchy–Schwartz inequalities on the right hand side of (5.21) must

$$\|\nabla\cdot\beta\|^2 + \|\theta_t\|^2 \le \|\delta_t\|^2 + \|\xi\|^2 + \|\beta\|^2. \tag{5.22}$$

replacing (5.22) into (5.20) we get

$$\|\theta\|^2 \le c\int_0^t (\|\xi\|^2+\|\delta_t\|^2+\|\beta\|^2)d\tau, \tag{5.23}$$

Then, using (5.23) into (5.16) we have

$$\|\zeta\|^2 \le C_3\int_0^t (\|\xi\|^2+\|\delta_t\|^2+\|\beta\|^2)d\tau, \tag{5.24}$$

Where $C_3 = C_3(c_0,c)$,
putting (5.23) and (5.24) into (5.14) we get

$$\|\beta\|^2+\|\xi\|^2 \le C_4\int_0^t (\|\eta\|^2+\|\delta_t\|^2+\|\beta\|^2+\|\delta\|^2+\|\xi\|^2)d\tau, \tag{5.25}$$

Where $C_4 = C_4(c,C_1,C_2)$.
Employing Gronwell's lemma, we have

$$\|\beta\|^2 + \|\xi\|^2 \le C_4 \int_0^t (\|\eta\|^2 + \|\delta_t\|^2 + \|\delta\|^2) d\tau, \tag{5.26}$$

Hence are estimates of β and ξ, replacing the the estimates in (5.23) to get the estimate of θ

$$\|\theta\|^2 \le C_5 \int_0^t (\|\eta\|^2 + \|\delta\|^2 + \|\delta_t\|^2) d\tau, \tag{5.27}$$

Where, $C_5 = C_5(c, C_4)$.
setting (5.23) into (5.15), to have the estimates of $\zeta, \nabla\zeta$,

$$\|\nabla\zeta\|^2 \le C_5 \int_0^t (\|\eta\|^2 + \|\delta\|^2 + \|\delta_t\|^2) d\tau, \tag{5.28}$$

Follow from (5.18) that

$$\|\zeta\|^2 \le C_5 \int_0^t (\|\eta\|^2 + \|\delta\|^2 + \|\delta_t\|^2) d\tau. \tag{5.29}$$

Thus, replacing (5.26) to (5.22)

$$\|\nabla\cdot\beta\|^2 + \|\theta_t\|^2 \le \|\delta_t\|^2 + C_4 \int_0^t (\|\eta\|^2 + \|\delta_t\|^2 + \|\delta\|^2) d\tau, \tag{5.30}$$

Finally, we calculate the required data from our theorem ,

$$\begin{aligned} \|u - u_h\|_1 = \|u - \Pi_h u + \Pi_h u - u_h\|_1 &\le \|\nabla(u - \Pi_h u)\| + \|\nabla(\Pi_h u - u_h)\| \\ &\le ch^m \|u\|_{m+1} + Ch^{\min(k+1,m)} \\ &\le Ch^{\min(k+1,m)}. \end{aligned} \tag{5.31}$$

Thus, the first requirement of the theorem has been proven.
Remark: placing (4.4) and (4.7) into (5.28) results

$$\|\nabla\zeta\|^2 \le C_5 h^{2\min(k+1,m)} \left(\|\boldsymbol{\sigma_t}\|^2_{L^\infty(H^{K+1})} + \|\boldsymbol{\sigma}\|^2_{L^\infty(H^{K+1})} + \|u\|^2_{L^\infty(H^{m+1})} \right),$$

$$\|\nabla\zeta\| \le Ch^{\min(k+1,m)},$$

C relies on C_5 $\|u\|_{m+1}$, and $\|\boldsymbol{\sigma_t}\|^2_{L^\infty(H^{K+1})}, \|\boldsymbol{\sigma}\|^2_{L^\infty(H^{K+1})}$ and $\|u\|^2_{L^\infty(H^{m+1})}$.
In the same way, we can prove what remains of the required theorem.
Particularly, utilizing (4.4) with (5.26) to make full the theorem (5.1(b)), thus (4.5) and (5.30) to have (c) from the Theorem 5.1, therefore the full evidence by (4.2), (4.4) and (4.7) with (5.30).

6. Fully-discrete and error estimates

In this part, the error estimates concerns with fully discrete. We well depend on backward Euler method, take $0 = t_0 < t_1 < \cdots < t_n < \cdots < t_M = T$ with $\Delta t = t_n - t_{n-1}, n = 1,2,\dots,M$ stands for the time grid and $\Delta t = T/M$, for some plus integer M, and put $t_n = n\Delta t$. For a smooth function $\Delta t\phi$ on $[0,T]$, define

$$\phi^n = \phi(t^n), \qquad \partial_t\phi = \frac{\phi^n - \phi^{n-1}}{\Delta t}. \tag{6.1}$$

Regarding approximate the integral term, we employ the right rectangle quadrature rule

$$q^n(\phi) = \Delta t \sum_{j=0}^{n-1} k_{n-j}\, \phi^j \approx \int_0^{t_n} k(t_n - s)\phi(s) ds, \tag{6.2}$$

where $k_{n-j} = k(t_n - s)$. This quadrature rule is positive [22,23], particularly.

$$\sum_{n=1}^{J} q^n(\phi)\, \phi^n = \Delta t \sum_{n=1}^{J} \sum_{j=1}^{n} k_{n-j}\, \phi^j\, \phi^n \ge 0, J = 1,\dots,M, \tag{6.3}$$

The quadrature error

$$R^n(\phi) = q^n(\phi) - \int_0^{t_n} k(t_n - s)\phi(s) ds, \tag{6.4}$$

holds

$$|R^n(\phi)| \le C\Delta t \int_0^{t_n} (|\phi(s)| + |\phi_t(s)|)ds, \tag{6.5}$$

Where $k, \phi \in C^1[0,T]$.

The equation (2.3) can be write as follows:

(a) $(\partial_t \boldsymbol{\sigma}^n, \boldsymbol{w}) + (\nabla\cdot \boldsymbol{q}^n, \nabla\cdot \boldsymbol{w}) - (\boldsymbol{p}^n, \nabla\cdot \boldsymbol{w}) = -(f^n, \nabla\cdot \boldsymbol{w}) + (R_1^n, \boldsymbol{w}),$ $\quad \forall \boldsymbol{w} \in W,$

(b) $(\boldsymbol{\sigma}^n, \nabla v) = (\nabla u^n, \nabla v),$ $\quad \forall v \in H_0^1(\Omega),$

$$(c)\ (\boldsymbol{q}^n, \boldsymbol{z}) - (\boldsymbol{\sigma}^n, \boldsymbol{z}) + \left(\Delta t \sum_{j=0}^{n-1} k_{n-j}\, a\left(u(t_j)\right) \boldsymbol{\sigma}^j, \boldsymbol{z}\right) + \left(\Delta t \sum_{j=0}^{n-1} k_{n-j}\, b\left(u(t_j)\right), \boldsymbol{z}\right)$$

$$= (-R_2^n + R_3^n, \boldsymbol{z}), \forall \boldsymbol{z} \in W, \tag{6.6}$$

$$(d)\ (\boldsymbol{p}^n, \boldsymbol{r}) - \left(\Delta t \sum_{j=0}^{n-1} k_{n-j}\, c\left(u(t_j)\right)\cdot \boldsymbol{\sigma}^j, \boldsymbol{r}\right) + \left(\Delta t \sum_{j=0}^{n-1} k_{n-j}\, g\left(u(t_j)\right), \boldsymbol{r}\right) = (R_4^n + R_5^n, \boldsymbol{r}), \quad \forall \boldsymbol{r} \in W.$$

Where

$$R_1^n = \partial_t \sigma^n - \sigma_t = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} (t_n - s) u_{tt} ds$$

$$R_2^n = \Delta t \sum_{j=0}^{n-1} k_{n-j}\, a\left(u(t_j)\right) \boldsymbol{\sigma}^j - \int_0^{t_n} k(t-s) a(u) \boldsymbol{\sigma}(\tau) d\tau = C\Delta t \int_0^{t_n} (|\boldsymbol{\sigma}(\tau)| + |\boldsymbol{\sigma}_t(\tau)|) d\tau,$$

$$R_3^n = \Delta t \sum_{j=0}^{n-1} k_{n-j}\, b\left(u(t_j)\right) - \int_0^{t_n} k(t-s) b(u) d\tau,$$

$$R_4^n = \Delta t \sum_{j=0}^{n-1} k_{n-j}\, c\left(u(t_j)\right)\cdot \boldsymbol{\sigma}^j - \int_0^{t_n} k(t-s) c(u)\cdot \boldsymbol{\sigma}(\tau) d\tau = C\Delta t \int_0^{t_n} (|\boldsymbol{\sigma}(\tau)| + |\boldsymbol{\sigma}_t(\tau)|) d\tau\,,$$

$$R_5^n = \Delta t \sum_{j=0}^{n-1} k_{n-j}\, g\left(u(t_j)\right) - \int_0^{t_n} k(t-s) g(u) d\tau,$$

Claim:

From the integral form of the remainder for the Taylor series of **f(x),**

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{1}{n!}\int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n dt$$

Then

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j + \frac{1}{n!}\int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n dt$$

And

$$f(x) - \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j = \frac{1}{n!}\int_{t=a}^{x} f^{(n+1)}(t)(x-t)^n dt.$$

Then, using the above equations as shown below:

$$u^{n-1} = u^n + \Delta t \partial_t u + \frac{\Delta t}{2!}\partial_{tt} u + \cdots + \sum_{j=0}^{n} \frac{\Delta t}{j!} \partial^{(n)} u + \frac{\Delta t}{n!}\int_t^x \partial^{(n+1)} u\, dt$$

There is four,

$$\partial_t \sigma^n - \sigma_t = \frac{\sigma^n - \sigma^{n-1}}{\Delta t} - \sigma_t = \frac{1}{\Delta t}(\sigma^n - \sigma^{n-1}) - \sigma_t$$

$$=\frac{1}{\Delta t}\left(\sigma^n-\left(\sigma^n-\Delta t\partial_t\sigma-\frac{1}{1!}\int_{t_{n-1}}^{t_n}\partial_{tt}\sigma(t_n-\tau)d\tau\right)\right)-\sigma_t$$
$$=\frac{1}{\Delta t}\left(\sigma^n-\sigma^n+\Delta t\partial_t\sigma-\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\partial_{tt}\sigma(t_n-\tau)d\tau\right)-\sigma_t$$
$$=\sigma_t-\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\partial_{tt}\sigma(t_n-\tau)d\tau-\sigma_t=-\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\partial_{tt}\sigma(t_n-\tau)d\tau$$
$$=\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\partial_{tt}\sigma(t_{n-1}-\tau)d\tau=R_1^n,$$

Also, by the quadrature error then (6.4) and (6.5) we get

$$R_2^n=\Delta t\sum_{j=0}^{n-1}k_{n-j}\,a(u^j)\boldsymbol{\sigma}^j-\int_0^{t_n}k(t-s)a(u)\boldsymbol{\sigma}(\tau)d\tau=C\Delta t\int_0^{t_n}(|\boldsymbol{\sigma}(\tau)|+|\boldsymbol{\sigma}_t(\tau)|)d\tau,$$
$$|R_2^n|\le C\Delta t\int_0^{t_n}(|\boldsymbol{\sigma}(\tau)|+|\boldsymbol{\sigma}_t(\tau)|)d\tau,$$

So that

$$|R_3^n|\le C\Delta t\int_0^{t_n}(|\boldsymbol{\sigma}(\tau)|+|\boldsymbol{\sigma}_t(\tau)|)d\tau,$$

Complement of the claim.

Then, we give a complete discrete way: figure out $(u_h^n,\boldsymbol{\sigma}_h^n,\boldsymbol{q}_h^n,\boldsymbol{p}_h^n)\in V_h\times\boldsymbol{W}_h\times\boldsymbol{W}_h\times\boldsymbol{W}_h$, $(n=0,1,2,\dots,M-1)$, such that

$$
\begin{aligned}
&(a)\ (\partial_t\boldsymbol{\sigma}_h^n,\boldsymbol{w}_h)+(\nabla\cdot\boldsymbol{q}_h^n,\nabla\cdot\boldsymbol{w}_h)-(\boldsymbol{p}_h^n,\nabla\cdot\boldsymbol{w}_h)=-(f^n,\nabla\cdot\boldsymbol{w}_h), && \forall\boldsymbol{w}_h\in\boldsymbol{W}_h,\\
&(b)\ (\boldsymbol{\sigma}_h^n,\nabla v_h)=(\nabla u_h^n,\nabla v_h), && \forall v_h\in V_h,\\
&(c)\ (\boldsymbol{q}_h^n,\boldsymbol{z}_h)-(\boldsymbol{\sigma}_h^n,\boldsymbol{z}_h)+\left(\Delta t\sum_{j=0}^{n-1}k_{n-j}a\left(u_h(t_j)\right)\boldsymbol{\sigma}_h^j,\boldsymbol{z}_h\right)-\left(\Delta t\sum_{j=0}^{n-1}k_{n-j}b\left(u_h(t_j)\right),\boldsymbol{z}_h\right)\\
&\qquad=(-R_2^n+R_3^n,\boldsymbol{z}_h),\ \ \forall\boldsymbol{z}_h\in\boldsymbol{W}_h, && (6.7)\\
&(d)\ (\boldsymbol{p}_h^n,\boldsymbol{r}_h)-\left(\Delta t\sum_{j=0}^{n-1}k_{n-j}c\left(u_h(t_j)\right)\cdot\boldsymbol{\sigma}^j,\boldsymbol{r}_h\right)-\left(\Delta t\sum_{j=0}^{n-1}k_{n-j}g\left(u_h(t_j)\right),\boldsymbol{r}_h\right)\\
&\qquad=(R_4^n+R_5^n,\boldsymbol{r}_h), && \forall\boldsymbol{r}_h\in\boldsymbol{W}_h.
\end{aligned}
$$

To selected the required error estimates, now we divide the errors,

$$\sigma(t_n)-\sigma_h^n=\sigma(t_n)-\mathrm{R}_h\sigma_h^n+\mathrm{R}_h\sigma_h^n-\sigma_h^n=\delta^n+\theta^n,$$
$$q(t_n)-q_h^n=q(t_n)-\mathrm{R}_hq_h^n+\mathrm{R}_hq_h^n-q_h^n=\alpha^n+\beta^n,$$
$$p(t_n)-p_h^n=p(t_n)-\mathrm{R}_hp_h^n+\mathrm{R}_hp_h^n-p_h^n=\rho^n+\xi^n,$$

and

$$u(t_n)-u_h^n=u(t_n)-\Pi_hu_h^n+\Pi_hu_h^n-u_h^n=\eta^n+\zeta^n,$$

putting (6.2) from (6.1) with (3.1) at $t=t_n$ we get the error equations below:

$$
\begin{aligned}
&(\boldsymbol{a})\ (\partial_t\theta^n,\boldsymbol{w}_h)+(\nabla\cdot\beta^n,\nabla\cdot\boldsymbol{w}_h)-(\xi^n,\nabla.\boldsymbol{w}_h)=(\partial_t\delta^n,\boldsymbol{w}_h)+(R_1^n,\boldsymbol{w}_h) && \forall\boldsymbol{w}_h\in\boldsymbol{W}_h,\\
&(\boldsymbol{b})\ (\theta^n,\nabla v_h)-(\nabla\zeta^n,\nabla v_h)=0, && \forall v_h\in V_h,\\
&(\boldsymbol{c})\ (\beta^n,\boldsymbol{z}_h)=(\theta^n,\boldsymbol{z}_h)-\left(\Delta t\sum_{j=0}^{n-1}k_{n-j}\left(a\left(u(t_j)\right)-a\left(u_h(t_j)\right)\right)\boldsymbol{\sigma}_h^j,\boldsymbol{z}_h\right)
\end{aligned}
$$

$$-\left(\Delta t\sum_{j=0}^{n-1}k_{n-j}\left(a\left(u(t_j)\right)\right)\delta^j,\boldsymbol{z}_h\right)-\left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(a\left(u(t_j)\right)\right)\theta^j,\boldsymbol{z}_h\right)$$
$$+\left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(b\left(u(t_j)\right)-b\left(u_h(t_j)\right)\right),\boldsymbol{z}_h\right)-(R_2^n+R_3^n,\boldsymbol{z}_h),\forall\boldsymbol{z}_h\in\boldsymbol{W}_h,\quad(6.8)$$

$$(\boldsymbol{d})\ (\xi^n,\boldsymbol{r}_h)=\left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(c\left(u(t_j)\right)-c\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\sigma}_h^j,\boldsymbol{r}_h\right)+\left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(c\left(u(t_j)\right)\right)\cdot\delta^j,\boldsymbol{r}_h\right)$$
$$+\left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(c\left(u(t_j)\right)\right)\cdot\theta^j,\boldsymbol{r}_h\right)+\left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(g\left(u(t_j)\right)-g\left(u_h(t_j)\right)\right),\boldsymbol{r}_h\right)$$
$$+(R_4^n+R_5^n,\boldsymbol{r}_h),\forall\boldsymbol{r}_h\in\boldsymbol{W}_h,$$

Theorem 6.1 let $\boldsymbol{\sigma}_h(0)=R_h\nabla u_0(x)$ and there is constant C independent of $h,\Delta t$ and $\|u\|_{m+1}+\|\sigma_t\|_{k+1}+\|p\|_{k+1},\|q\|_{k+1}$, like

$$(a)\quad\left\|u(t_J)-u_h^J\right\|_1\le C\left(h^{min(m,k+1)}+\int_o^{t_n}\|u_{tt}\|d\tau+\int_o^{t_n}(\|\sigma(\tau)\|+\|\sigma_t(\tau)\|)d\tau\right)+\Delta t$$

$$(b)\quad\left\|\boldsymbol{p}(t_J)-\boldsymbol{p}_h^J\right\|\le C\left(h^{min(m+1,k+1)}+\int_o^{t_n}\|u_{tt}\|d\tau+\int_o^{t_n}(\|\sigma(\tau)\|+\|\sigma_t(\tau)\|)d\tau\right)+\Delta t$$

$$(c)\quad\left\|\nabla\cdot\left(\boldsymbol{q}(t_J)-\boldsymbol{q}_h^J\right)\right\|\le C\left(h^{min(m+1,k)}+\int_o^{t_n}\|u_{tt}\|d\tau+\int_o^{t_n}(\|\sigma(\tau)\|+\|\sigma_t(\tau)\|)d\tau\right)+\Delta t$$

$$(d)\quad\left\|u(t_J)-u_h^J\right\|+\left\|\boldsymbol{\sigma}(t_J)-\boldsymbol{\sigma}_h^J\right\|+\left\|\boldsymbol{q}(t_J)-\boldsymbol{q}_h^J\right\|+\left\|\boldsymbol{p}(t_J)-\boldsymbol{p}_h^J\right\|$$
$$\le C\left(h^{min(m+1,k+1)}+\int_o^{t_n}\|u_{tt}\|d\tau+\int_o^{t_n}(\|\sigma(\tau)\|+\|\sigma_t(\tau)\|)d\tau\right)+\Delta t$$

Proof. Supposing $\boldsymbol{z}_h=\beta^n$ in (6.8(c)) ,

$$\|\beta^n\|^2=(\theta^n,\beta^n)-\left(\Delta t\sum_{j=0}^{n-1}k_{n-j}\left(a\left(u(t_j)\right)-a\left(u_h(t_j)\right)\right)\boldsymbol{\sigma}_h^j,\beta^n\right)$$
$$-\left(\Delta t\sum_{j=0}^{n-1}k_{n-j}\left(a\left(u(t_j)\right)\right)\delta^j,\beta^n\right)-\left(\Delta t\sum_{j=0}^{n-1}k_{n-j}\left(a\left(u(t_j)\right)\right)\theta^j,\beta^n\right)$$
$$+\left(\Delta t\sum_{j=0}^{n-1}k_{n-j}\left(b\left(u(t_j)\right)-b\left(u_h(t_j)\right)\right),\beta^n\right)-(R_2^n+R_3^n,\beta^n),\quad(6.9)$$

employing Cauchy-Schwartz inequalities for every expression on the right hand side of the above equation to have

$$|(\theta^n,\beta^n)|\le\|\theta^n\|\|\beta^n\|,\quad(6.10)$$

$$\left|-\left(\sum_{j=0}^{n-1}k_{n-j}\left(a\left(u(t_j)\right)-a\left(u_h(t_j)\right)\right)\boldsymbol{\sigma}_h^j,,\beta^n\right)\right|\le cc_1c_2\left(\sum_{j=0}^{n-1}\left(\left\|\eta^j\right\|+\left\|\zeta^j\right\|\right)\right)\|\beta^n\|,\quad(6.11)$$

where, c_1 relies on k_{n-j}, c_2 relies on $\left\|\boldsymbol{\sigma}^j\right\|_{W_\infty^1(L^\infty)}$, $a\left(u(t_j)\right)$ is Lipchitz constant concerning u,

$$\left|-\left(\sum_{j=0}^{n-1}k_{n-j}\left(a\left(u(t_j)\right)\right)\delta^j,\beta^n\right)\right|\le cc_1c_3\sum_{j=0}^{n-1}\left\|\delta^j\right\|\|\beta^n\|,\quad(6.12)$$

c_3 depends on the bound of $a\left(u(t_j)\right)$,

$$\left|-\left(\sum_{j=0}^{n-1} k_{n-j}\left(a\left(u(t_j)\right)\right)\theta^j, \beta^n\right)\right| \le cc_1c_3\sum_{j=0}^{n-1}\|\theta^j\|\|\beta^n\|, \tag{6.13}$$

$$\left|\left(\sum_{j=0}^{n-1} k_{n-j}\left(b\left(u(t_j)\right)-b\left(u_h(t_j)\right)\right), \beta^n\right)\right| \le cc_1\sum_{j=0}^{n-1}(\|\eta^j\|+\|\zeta^j\|)\|\beta^n\|, \tag{6.14}$$

where $b\left(u(t_j)\right)$ is Lipchitz fixed regarding u,

$$|-(R_2^n,\beta^n)| \le \|R_2^n\|^2+\varepsilon\|\beta^n\|^2 \le \frac{c_4}{\Delta t}\left(\int_{t_{n-1}}^{t_n}\|u_{tt}(\tau)\|d\tau\right)\|\beta^n\|. \tag{6.15}$$

then, in the light of (6.2), we have

$$|-(R_3^n,\beta^n)| = \left|\left(\Delta t\sum_{j=0}^{n-1} k_{n-j}\, b(u^j), \beta^n\right)-\left(\int_0^{t_n} k(t-s)b(u)d\tau, \beta^n\right)\right| \lesssim \Delta t\|\beta^n\|, \tag{6.16}$$

Replacing (6.10)-(6.16) into (6.9) we have Type equation here.

$$\|\beta^n\|^2 \le C_1\left(\|\theta^n\|+\left(\sum_{j=0}^{n-1}(\|\eta^j\|+\|\zeta^j\|)\right)+\sum_{j=0}^{n-1}\|\delta^j\|+\sum_{j=0}^{n-1}\|\theta^j\|+\sum_{j=0}^{n-1}(\|\eta^j\|+\|\zeta^j\|)\right.$$
$$\left.+\frac{1}{\Delta t}\left(\int_{t_{n-1}}^{t_n}\|u_{tt}(\tau)\|d\tau\right)+\Delta t\right)\|\beta^n\|, \tag{6.17}$$

where $C_1 = C_1(c, c_1, c_2, c_3, c_4)$,

Multiplying by Δt, as well as summing (6.17) from $n = 1$ to J, and in the light of (6.3), the second, third, fourth and fifth terms on the left hand side are nonnegative, we have

$$\Delta t\|\beta^J\| \le C_1\left(\Delta t\|\theta^J\|+\Delta t\int_{t_{n-1}}^{t_n}\|u_{tt}(\tau)\|d\tau+(\Delta t)^2\right), \tag{6.18}$$

Then,

$$\|\beta^J\| \le C\left(\|\theta^J\|+\int_{t_{n-1}}^{t_n}\|u_{tt}(\tau)\|d\tau+\Delta t\right). \tag{6.19}$$

Here find the estimate of ξ^n ,we selecting $\boldsymbol{r}_h = \xi^n$ in (6.3(d)) we have

$$\|\xi^n\|^2 = \left(\sum_{j=0}^{n-1} k_{n-j}\left(c\left(u(t_j)\right)-c\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\sigma}_h^j, \xi^n\right)+\left(\sum_{j=0}^{n-1} k_{n-j}\left(c\left(u(t_j)\right)\right)\cdot\delta^j, \xi^n\right)$$
$$+\left(\Delta t\sum_{j=0}^{n-1} k_{n-j}\left(c\left(u(t_j)\right)\right)\cdot\theta^j, \xi^n\right)+\left(\Delta t\sum_{j=0}^{n-1} k_{n-j}\left(g\left(u(t_j)\right)-g\left(u_h(t_j)\right)\right), \xi^n\right)$$
$$+(R_4^n+R_5^n,\xi^n), \tag{6.20}$$

Employing Cauchy-Schwartz inequalities for every expression on the right hand side of the equation to have:

$$\left|\left(\sum_{j=0}^{n-1} k_{n-j}\left(c\left(u(t_j)\right)-c\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\sigma}_{h'}^j, \xi^n\right)\right| \le c_1c_2\left(\sum_{j=0}^{n-1}(\|\eta^j\|+\|\zeta^j\|)\right)\|\xi^n\|, \tag{6.21}$$

where $c\left(u(t_j)\right)$ is Lipchitz continuous regarding u,

$$\left|\left(\sum_{j=0}^{n-1} k_{n-j}\left(c\left(u(t_j)\right)\right)\cdot\delta^j, \xi^n\right)\right| \le c_1c_4\sum_{j=0}^{n-1}\|\delta^j\|\|\xi^n\|, \tag{6.22}$$

c_4 depend on the bound of $c\left(u(t_j)\right)$,

$$\left|\left(\sum_{j=0}^{n-1} k_{n-j}\left(c\left(u(t_j)\right)\right)\cdot\theta^j,\xi^n\right)\right| \le c_1 c_4 \sum_{j=0}^{n-1}\|\theta^j\|\|\xi^n\|, \tag{6.23}$$

$$\left|\left(\sum_{j=0}^{n-1} k_{n-j}\left(g\left(u(t_j)\right)-g\left(u_h(t_j)\right)\right),\xi^n\right)\right| \le c_1\left(\sum_{j=0}^{n-1}(\|\eta^j\|+\|\zeta^j\|)\right)\|\xi^n\|, \tag{6.24}$$

where $g\left(u(t_j)\right)$ is Lipchitz fixed concerning u,
therefore, in view of (6.5) we get:

$$|(R_4^n,\xi^n)| \le C\Delta t\left(\int_0^t (\|\sigma(\tau)\|+\|\sigma_t(\tau)\|)d\tau\right)\|\xi^n\|, \tag{6.25}$$

Now, in view of (6.2), we have:

$$|(R_5^n,\xi^n)| = \left|\left(\Delta t\sum_{j=0}^{n-1} k_{n-j}\, g\left(u(t_j)\right),\xi^n\right)-\left(\int_0^{t_n} k(t-s)g(u)d\tau,\xi^n\right)\right| \lesssim \Delta t\|\xi^n\|, \tag{6.26}$$

Replacing (6.22)-(6.27) into (6.21) we have:

$$\|\xi^n\|^2 \le c_1c_2\left(\sum_{j=0}^{n-1}(\|\eta^j\|+\|\zeta^j\|)\right)\|\xi^n\| + c_1c_4\sum_{j=0}^{n-1}\|\theta^j\|\|\xi^n\| + c_1\left(\sum_{j=0}^{n-1}(\|\eta^j\|+\|\zeta^j\|)\right)\|\xi^n\|$$
$$+c_1c_4\sum_{j=0}^{n-1}\|\delta^j\|\|\xi^n\| + C\Delta t\left(\int_0^t(\|\sigma(\tau)\|+\|\sigma_t(\tau)\|)d\tau\right)\|\xi^n\| + \Delta t\|\xi^n\|, \tag{6.27}$$

According to (6.28) from $n=1$ to J, as well as in the light of (6.3), the first, second, third and fourth terms on the left hand side are nonnegative, we have

$$\|\xi^J\| \le C\left(\int_0^t(\|\sigma(\tau)\|+\|\sigma_t(\tau)\|)d\tau+\Delta t\right), \tag{6.28}$$

Then, we estimate θ^J, select $w_h=\theta^n$ in (6.8(a)) as well as use the Cauchy-Schwarz inequality and Young's inequality to have:

$$\frac{1}{2}\partial_t\|\theta^n\|^2 \le C(\|\partial_t\delta^n\|^2+\|\xi^n\|^2+\|\nabla\cdot\beta^n\|^2+\|R_1^n\|^2)+C(\|\theta^n\|^2+\|\nabla\cdot\theta^n\|^2), \tag{6.29}$$

Note that

$$\Delta t\sum_{n=1}^{J}\|\partial_t\delta^n\|^2 \le Ch^{2(k+1)}\|\boldsymbol{\sigma_t}\|_{k+1}^2 \tag{6.30}$$

$$\Delta t\sum_{n=1}^{J}\|R_1^n\|^2 \le C\Delta t\int_0^{t_J}\|u_{tt}(\tau)\|^2\,d\tau, \tag{6.31}$$

on replacing (6.30) and (6.31) into (6.29) as well as summing it from $n=1$ to J, we obtain:

$$\|\theta^J\|^2 \le Ch^{2(k+1)}\|\boldsymbol{\sigma_t}\|_{k+1}^2+\sum_{n=1}^{J}\|\xi^n\|^2+\sum_{n=1}^{J}\|\nabla\cdot\beta^n\|^2+C\Delta t\int_0^{t_J}\|u_{tt}(\tau)\|^2d\tau+\sum_{n=1}^{J}C(\|\theta^n\|^2+\|\nabla\cdot\theta^n\|^2),$$

Use Gronwell's lemma to get

$$\|\theta^J\|^2 \le Ch^{2(k+1)}\|\boldsymbol{\sigma_t}\|_{k+1}^2+\sum_{n=1}^{J}\|\xi^n\|^2+\sum_{n=1}^{J}\|\nabla\cdot\beta^n\|^2+C\Delta t\int_0^{t_J}\|u_{tt}(\tau)\|^2d\tau, \tag{6.32}$$

combining (6.28) and (6.32) and also applying Gronwell's lemma to obtain

$$\|\xi^J\|+\|\theta^J\| \le C\left(h^{k+1}\|\boldsymbol{\sigma_t}\|_{k+1}+\Delta t\int_0^{t_J}\|u_{tt}(\tau)\|d\tau+\int_0^{t_n}(\|\sigma(\tau)\|+\|\sigma_t(\tau)\|)d\tau+\Delta t\right), \tag{6.33}$$

Now putting the estimate of θ^J into (6.33) we get the estimate of β^J,

$$\|\beta^J\| \le C\left(h^{k+1}\|\boldsymbol{\sigma_t}\|_{k+1} + \Delta t\int_0^{t_n}\|u_{tt}(\tau)\|d\tau + + \int_0^{t_n}(\|\sigma(\tau)\| + \|\sigma_t(\tau)\|)d\tau + \Delta t\right). \quad (6.34)$$

Taking $v_h = \zeta^n$ in (6.8(b)), we have

$$\|\nabla\zeta^n\| \le \|\theta^n\|, \quad (6.35)$$

by Poincare inequality, we obtain

$$\|\zeta^n\| \le \|\nabla\zeta^n\| \le \|\theta^n\|, \quad (6.36)$$

then

$$\|\zeta^n\| \le C\left(h^{k+1}\|\boldsymbol{\sigma_t}\|_{k+1} + \Delta t\int_0^{t_J}\|u_{tt}(\tau)\|d\tau + \int_0^{t_n}(\|\sigma(\tau)\| + \|\sigma_t(\tau)\|)d\tau + \Delta t\right). \quad (6.37)$$

After that using of the triangle inequality with (6.37), (6.33), (6.34), (4.5), (4.4), and (4.7) complete the proof.

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