Research Article

Study and analyze the eigenvalues and eigenvectors of a square matrix and study their applications through mathematical linear effects

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Article History: Received: 10 January 2021; Revised: 12 February 2021; Accepted: 27 March 2021; Published online: 20 April 2021

Abstract: In this paper we will investigate what properties are intrinsic to a matrix, or its associated linear application. As we will see, the fact that there are many bases in a vector space makes the expression of matrices or linear applications relative: it depends on which reference base we take. However, there are elements associated with this matrix, which do not depend on the reference base or bases that we choose, for example: a null spaces and a column spaces of a matrix, and their respective dimensions. The eigen values of matrix isa root of a character polynomial. Find a eigenvalue of matrix is equivalent to finding a rootof its polynomial. For matrices of size n ≥ 5, there does not generally existclosed expressions for the roots of the characteristic polynomial based on primary expressions (additions, subtractions, multiplications, divisions and roots). This result implies that the methods to find the valuesof a matrix must be iterativeOne way to calculate the eigen values would be to calculate the roots of thecharacteristic polynomial using a numerical method of calculating roots, like roots in Matlab. Roots in Python. But find the roots of a polynomial is usually a poorly conditioned problem. The conditioning of a problem has not been defined, but a wrong problemconditioned is a problem for which a small change in the data can induce an uncontrolled change in the results.

Keywords: mathematical linear effects, matrix, eigen values

1. Eigenvectors and Eigenvalues

In this paper we will focus on square matrices, with respective matrix applications that can be interpreted as transformations of the R^n space. The application x → Ax can transform a given vector x into another, of different direction and length. However, there may be some very special vectors for that transformation. For example, the set of vectors that are transformed to zero by the application, that is, the nucleus of the application or null space of A: {x: Ax = 0}. As much as we change the base, the null space is always the same. Another interesting set is that of the vectors that transform themselves, {x: Ax = x}. The two commented sets are associated with the matrix, and have a special characteristic: the transformation keeps them invariant, that is, the transformed vectors fall within the corresponding sets. As much as we change bases, the sets remain invariant. Give an example of a non-invariant set: a line that is transformed into another with a certain angle.

See A=$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ **,u=** $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ **and v=**$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.Result that

Au= $\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, Av= $\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ =2v.

The application has transformed v without changing its direction.

Vectors that are only stretched or shrunk by a linear application are very special to it.

2. Definition An eigenvector (or eigenvector) of a matrix A of n× n is a vector x ∈ R^n, other than 0, likethe a certain scalar$\boldsymbol{\lambda}$∈R

Ax = $\boldsymbol{\lambda}$x.

A scalar $\boldsymbol{\lambda}$such is called the eigenvalue (or eigenvalue) of A, that is, $\boldsymbol{\lambda}$it is the eigenvalue of A if there is a nontrivial solution of Ax =$\boldsymbol{\lambda}$ x, and x is called the eigenvector associated with the eigenvalue$\boldsymbol{\lambda}$.

Be A= $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

1. Check if **u**=$\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and v= $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ are eigenvectors of A.

2. Determine if $\boldsymbol{\lambda}$ = 7 is the eigenvalue of A.

Solution:

1. You have to determine if the equation

$$Ax = 7x \Leftrightarrow (A-7I)x = 0$$

(which we have written as a homogeneous system) has a solutionnot trivial. To do this write

A- 7I =$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

The associated homogeneous system has a free variable, thenthere are nontrivial solvewith 7 is a eigenvalue of A. In fact, a solution of the homogeneous system is $x_1 = x_2$, that is, in parametric vector form x = $x_1\begin{bmatrix}\mathbf{1}\\ \mathbf{1}\end{bmatrix}$these infinite vectors (minus 0) are the eigenvectors associated with the eigenvalue 7.

Summarizing, if for a given matrix A and a scalar $\boldsymbol{\lambda}$there are nontrivial solutions of the homogeneous equation

$$(A-\boldsymbol{\lambda}I)x=0$$

Then$\boldsymbol{\lambda}$it is an eigenvalue of A, and the set of all solutions is the set of eigenvectors associated with$\boldsymbol{\lambda}$ (plus 0), with call a proper space of A associated with$\boldsymbol{\lambda}$. Or in another word, a proper space associated$\boldsymbol{\lambda}$ with is the Nul space (A- $\boldsymbol{\lambda}$ I), if it is not null.

Prove that a proper space is a vector subspace.

The space of the vectors that a matrix transforms into itself is its own space. Which?

Be A= $\begin{bmatrix}4 & -1 & 6\\ 2 & 1 & 6\\ 2 & -1 & 8\end{bmatrix}$. Check that 2 are an eigenvalue, and find a base of the associated proper space.

Solution: Considering the matrix

$$A\text{-}2I=\begin{bmatrix}2 & -1 & 6\\ 2 & -1 & 6\\ 2 & -1 & 6\end{bmatrix}\sim\begin{bmatrix}2 & -1 & 6\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

With which the own space is based

$$\left\{\begin{bmatrix}1/2\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-3\\ 0\\ 1\end{bmatrix}\right\}$$

And it is two-dimensional. How does the matrix transformation associated with A act on this proper space?

The proper space of an array A associated with the value$\boldsymbol{\lambda}$= 0 has another name.

If 0 is the eigenvalue of a matrix A, if with just if A is non-invertible.

Theorem. let v_1, ..., v_r eigenvector corresponding to different eigenvalue $\boldsymbol{\lambda}_1$, ...,$\boldsymbol{\lambda}_r$ of a matrix A n× n, then agroup $\{v_1, \dots, v_r\}$ is linearly independent.

Demonstration. Suppose that the vector set is linearly dependent. There will be a first vector v_{p+1} that will be a linear combination of the previous ones (theorem 3.20)

$$v_{p+1} = c_1 v_1 + \dots + c_p v_p. \qquad (1)$$

Multiplying by A

$Av_{p+1} = c_1 Av_1 + \dots + c_p Av_p \Rightarrow_{p+1} v_{p+1} = c_1 \boldsymbol{\lambda}_1 v_1 + \dots + c_p \boldsymbol{\lambda}_p v_p$

Subtracting $\boldsymbol{\lambda}_{p+1}$ times (6.1) from this last equation.

$$C_1 (\boldsymbol{\lambda}_1 - {}_{p+1})\, v_1 + \dots + c_p (\boldsymbol{\lambda}_p - \boldsymbol{\lambda}_{p+1})\, v_p = 0.$$

But $\{ v_1, \dots v_p\}$ is linearly independent, and $\boldsymbol{\lambda}_i$-$\boldsymbol{\lambda}_{p+1} \neq 0$ if i <p + 1, so $c_1 = \dots = c_p = 0$. This is a contradiction, and $\{v_1, \dots v_r\}$must be linearly independent.

Show that an n× n matrix cannot have more than one distinct eigenvalues.

2. A characteristic equations

Example. Find the eigenvalues of **A=**$\begin{bmatrix}\mathbf{2} & \mathbf{3}\\ \mathbf{3} & \mathbf{-6}\end{bmatrix}$ **. Solution**

Theorem (Extension of the invertible matrix theorem). Let A n× n. Then it is equivalent to A being invertible any of the following statements.

s. 0 is not an eigenvalue of A.

t. The determinant of A is not zero.

What was said in the previous section shows yes. And Theorem 5.11 shows t.

The characteristic equation.

Proposition. A scalar is $\boldsymbol{\lambda}$an eigenvalue of a square matrix A if and only if it satisfies the so-called characteristic equation

$$\mathbf{Det\ (\ A - \boldsymbol{\lambda} I)=0\ .}$$

Example. Characteristic equation of

$$\mathbf{A}=\begin{bmatrix}\mathbf{5} & \mathbf{-2} & \mathbf{6} & \mathbf{-1}\\ \mathbf{0} & \mathbf{3} & \mathbf{-8} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{5} & \mathbf{4}\end{bmatrix}$$
$$\mathbf{0\quad 0\quad 0\quad 1}$$

It is clear that the characteristic equation is equal to a polynomial equation, since given a numerical matrix A of n × n, the expression det (A−$\boldsymbol{\lambda}$I) is a polynomial in. This polynomial, which is of degree n, is called the characteristic polynomial of matrix A. Clearly, its roots are the eigenvalues of A.

The algebraic multiplicity of an eigenvalue is its multiplicity as the root of the characteristic polynomial. That is, we know that a polynomial with complex coefficients can always be factored into simple factors:

$$\det(A-\lambda I) = (\lambda_1 - \lambda)^{\mu 1}(\lambda_2 - \lambda)^{\mu 2} \ldots (\lambda_r - \lambda)^{\mu r}\ .$$

The roots λ_i are the eigenvalues, and the exponents μ_i are the algebraic multiplicities of the corresponding eigenvalues λ_i.

Theorem. The eigenvalues of a triangular matrix are the inputs of its main diagonal.

Example. The characteristic polynomial of a matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$.

1. How big is the matrix?
2. What are their eigenvalues and corresponding multiplicities?

Likeness. The row reduction process has been our main tool for solving systems of equations. The fundamental characteristic of this procedure is, again, the invariance of the set of solutions; the object sought, with respect to the transformations used to simplify the system, the elementary operations by rows.

In the case that the objects of interest are the eigenvalues, there is a procedure that keeps them invariant and allows simplifying the matrix A.

Definition. Two matrices A and B of n × n are said to be similar if there is an invertible matrix P of n×n that relates them by the following formula

Demonstration. We know that $B = P^{-1}AP$. Thus

$B - \lambda I = B = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$.

Therefore, the characteristic polynomial of B

$\det(B-\lambda I) = \det[P^{-1}(A-\lambda I)P] = \det(P^{-1}) \det(A-\lambda I) \det(P) = \det(A-\lambda I)$

It is the same as that of A. Note that we have used that $\det(P^{-1}) = 1 / \det(P)^*$.

It is important to note that row equivalence is not the same as similarity. The row equivalence is written matrixally as B = EA, for a certain invertible matrix E; the similarity as $B = P^{-1}AP$ for a certain invertible matrix P.

3. Diagonalization

A very special class of square matrices is diagonal matrices, those whose elements are all null, except for the main diagonal:

$$D=\begin{bmatrix} d1 & 0 & \ldots & 0 \\ 0 & d2 & \ldots & 0 \\ 0 & 0 & \ldots & dn \end{bmatrix}$$

Its action on vectors is very simple.

Example. Be $D=\begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}$.So

$$D\begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}\begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 5x1 \\ 3x2 \end{bmatrix}.$$

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We can therefore deduce that $D^2= \begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}\begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{bmatrix} = \begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}$ and in general

k

$$D^k= \begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{bmatrix} = \begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}$$

Which is the form for the k-th power of D, and it naturally extends to any diagonal matrix.

The power of an array is very useful in many applications, as we will see later. In fact, we would like to compute the k-th power of any matrix. This is calculated very easily if we manage to diagonalize A, that is, find a diagonal matrix D similar to A: $A = PDP^{-1}$. The reason is very simple, as the following example illustrates.

Example. Be $A= \begin{bmatrix} \mathbf{7} & \mathbf{2} \\ \mathbf{-4} & \mathbf{1} \end{bmatrix}$. It can be verified that $A= PDP^{-1}$ with

$P= \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-2} \end{bmatrix}$ and $D=\begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}$(and $P^{-1} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-1} \end{bmatrix}$)

With this information, we find a formula for power k-thA^k of A. Square A^2 is

$A^2 = (PDP^{-1})(P|DP^{-1}) = PDP^{-1}P\ \ DP^{-1} = PDDP^{-1} = = PD^2P^{-1}$

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$$= \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-2} \end{bmatrix}\begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-2} \end{bmatrix}\begin{bmatrix} \mathbf{2.5} & \mathbf{5} \\ \mathbf{-3} & \mathbf{-3} \end{bmatrix}$$

$$=\begin{bmatrix} \mathbf{2.5-3} & \mathbf{5-3} \\ \mathbf{-2.5+2.3} & \mathbf{-5+2.3} \end{bmatrix}$$

It is easy to deduce that

$A^k = PDP^{-1}\, PDP^{-1k \text{ times in total}}\, PDP^{-1} = PD^kP^{-1}$

So

$A^k = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-2} \end{bmatrix} \begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{bmatrix} \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{2.5-3} & \mathbf{5-3} \\ \mathbf{-2.5+2.3} & \mathbf{-5+2.3} \end{bmatrix}$.

A matrix is diagonalizable if it can be diagonalized, that is, if there is a diagonal matrix D similar to A, so A = PDP $^{-1}$.

The k-th power of a matrix A that can be diagonalized $A = PDP^{-1}$ is $A^k = PD^kP^{-1}$.

$A = P\begin{bmatrix} \lambda 1 & & \\ & \lambda 2 & \\ & & \lambda n \end{bmatrix} P^{-1}, P = [v_1|\, v_2|\, ,,,\, |v_n]$, $Av_i = \boldsymbol{\lambda}_i v_i$.

Demonstration.

$AP = A\,[v_1|\, v_2|\, ,,,\, |v_n] = [A\,v_1|\, Av_2|\, ,,,\, |Av_n]$

$= [\boldsymbol{\lambda}_1 v_1|\boldsymbol{\lambda}_2 v_2|\, ,,,\, |\boldsymbol{\lambda}_n v_n] = \begin{bmatrix} \lambda 1 & & \\ & \lambda 2 & \\ & & \lambda n \end{bmatrix} D = PD$

$AP = PD \Rightarrow A = PDP^{-1}$

Because P is invertible as it is square and its columns are independent.

Viceversa, if $A = PDP^{-1}$ then $AP = PD$ and if v_i is column i of P, then this matrix equality implies that $Av_i = \boldsymbol{\lambda}_i v_i$, that is, the columns of P are eigenvectors. Being P invertible, they are linearly independent.

If a matrix does not have n linearly independent eigenvectors, it cannot be diagonalized.

Example. Let's diagnose

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Step 1. Find the eigenvalues of A. The characteristic equation is

$$\mathrm{Det}(A - \boldsymbol{\lambda} I) = \boldsymbol{\lambda}^3 - 3\boldsymbol{\lambda}^2 + 4 = -(\boldsymbol{\lambda} - 1)(\boldsymbol{\lambda}+2)^2 \,\mathbf{.}$$

Then there are two eigenvalues, $\boldsymbol{\lambda} = 1$ and $\boldsymbol{\lambda} = -2$ (with multiplicity 2).

Step 2.Find the eigenvector of A. Solving the systems $(A-\boldsymbol{\lambda} I)\, x = 0$ and giving the solution in vector form for etrica, we obtain bases of the eigen spaces:

$(A-I)\, x = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$(A+2\,I)x = 0 \Rightarrow v2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Note: if there are not three independent eigenvectors, the matrix is not can diagonalize.

Step 3. Build the matrix P. The columns of P are the eigenvectors:

$$P = [v_1|\, v_2|\, v_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 4. Construct matrix D. Place the diagonal of D on the eigenvalues, in the order corresponding to how the eigenvectors are placed in P (v_1 is the eigenvector with $\boldsymbol{\lambda}_1 = 1$, and v_2 and v_3 corresponda $\boldsymbol{\lambda}_2 = -2$:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The similarity can be verified by checking that AP = PD (for do not calculate P^{-1})

Is AP = PD fully equivalent to $A = PDP^{-1}$?

Is it possible to diagonalizable $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Theorem. The fact that A of n× n has n different eigenvalues is sufficient to ensure that it is diagonalizable.

Example. Is it diagonalizable $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$?

Demonstration.

Example. Diagonalizable yourself, if possible

$$\mathbf{A}=\begin{bmatrix} \mathbf{5} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{5} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{4} & \mathbf{-3} & \mathbf{0} \end{bmatrix}$$
$$\mathbf{-1} \quad \mathbf{-2} \quad \mathbf{0} \quad \mathbf{-3}$$

Example. Diagonalizable yourself, if possible

$$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5. Eigenvectors and linear transformations

5.1. Base change and linear transformations.

Theorem 4.18 assures us that any linear application T from R^n to R^m canbe implemented as a matrix application

$$T(x) = Ax \qquad (2)$$

Where A is the canonical matrix of T. This is a matrix m n that by columns was written with the action of the transformation on the vectors of the base A= [T (e_1) | … | T (e_n)]. Let's look first at the particular case n = m, whereby T it is a linear transformation of R^n and A is square. Let's denote

$$A= [T]_E = [\ T (e_1) \mid \dots \mid T (e_n)] \qquad (3)$$

Indicating that $[T]_E$ is the matrix of the application T that acts on vectors in canonical coordinates, and returns values as vectors also in the canonical base (later we will also write $[T]_E = [T]_{E \leftarrow E}$) The formula (6.2) is rewritten as

$$[T\,x]_E = A[x]_E = [T]_E[x]_E \ (4)$$

We want to understand how the action of the application would be on vectors coordinated on another base $\mathfrak{B} = \{b_1,\dots,b_n\}$ from R^n. Recall that the matrix of the coordinate change from $\mathfrak{B}$ to E is one whose columns are the vectors of $\mathfrak{B}$ written in the canonical base:

$$\mathbf{P_{\mathfrak{B}}=P_{\mathfrak{B}\leftarrow\mathfrak{B}}= [\ b_1 \mid b_2 \mid \dots \mid b_n]}$$

And the equation $x = c_1b_1 + c_2b_2 + \dots + c_nb_n$ is written in matrix form

$$[x]_E = P_{\mathfrak{B}}[x]_{\mathfrak{B}} \ [x]_{\mathfrak{B}} = (c_1, c_2,\dots,c_n)$$

($P_{\mathfrak{B}}$ "passes" coordinates in $\mathfrak{B}$ to coordinates in E) We would like to find a matrix B = $[T]_{\mathfrak{B}}$ that would act as T, but accepting vectors $[x]_{\mathfrak{B}}$in coordinates of $\mathfrak{B}$ and returning the resulting vector in base $\mathfrak{B}$ also:

$$[Tx]_{\mathfrak{B}} = B[x]_{\mathfrak{B}} = [T]_{\mathfrak{B}}[x]_{\mathfrak{B}} \quad (5)$$

Using the base change matrix $P_{\mathfrak{B}} = P_{E\leftarrow\mathfrak{B}}$ we can deduce how this matrix is. We can multiply $P_{\mathfrak{B}}\,[x]_{\mathfrak{B}} = [x]_E$ to pass it to canonical coordinates, and act on this vector with A, to obtain the vector $[T (x)]_E$:

$[x]_{\mathfrak{B}} \rightarrow [x]_E = P\,[x]_{\mathfrak{B}}$

$\rightarrow [T(x)]_E = A[x]_E = AP\,[x]_{\mathfrak{B}}$

Finally, to obtain the resulting vector $[T\,x]_{\mathfrak{B}}$ it is necessary to change the base$[T\,x]_E$ with the inverse matrix of the base change $P_{\mathfrak{B}\leftarrow E} = P^{-1}$

$\rightarrow [T(x)]_{\mathfrak{B}} = P_{\mathfrak{B}\leftarrow E}\,[T(x)]_E = P^{-1}AP\,[x]_{\mathfrak{B}}$

Writing it all together

$$[T\,x]_{\mathfrak{B}} = P_{\mathfrak{B}\leftarrow E}\,[T]_E P_{E\leftarrow\mathfrak{B}}\,[x]_{\mathfrak{B}}$$
$$[x]_E = P_{E\leftarrow\mathfrak{B}}[x]_{\mathfrak{B}}$$
$$[T\,x]_E = [T]_E\,[x]_E$$
$$[T\,x]_{\mathfrak{B}} = P_{\mathfrak{B}\leftarrow E}\,[T\,x]_E$$

We have deduced that

$$\mathbf{[T\,x]_{\mathfrak{B}} = P^{-1}AP\,[x]_{\mathfrak{B}}}$$

The formula (6.4) was $[T\,x]_{\mathfrak{B}} = B[x]_{\mathfrak{B}}$, so the matrix we were looking for is B= $[T]_{\mathfrak{B}} = P^{-1}AP$. We summarize with the following result.

$[T]_{\mathfrak{B}} = P^{-1}[T]_E P$ *siendo* $P = P_{E\leftarrow\mathfrak{B}} = P_{\mathfrak{B}}$ (6)

The matrix $[T]_{\mathfrak{B}} = [T]_{\mathfrak{B}\leftarrow\mathfrak{B}}$ is called the $\mathfrak{B}$ -matrix of T.

There is a direct way to calculate the $\mathfrak{B}$-matrix of T. Since $x = c_1b_1 + \dots + c_nb$ So

$T(x) = T(c_1b_1 + \_\ \_\ \_ + c_nb_n) = c_1T(b_1) + \_\ \_\ \_ + c_nT(b_n)$

is the resulting vector. If we want it in coordinates of $\mathfrak{B}$ we have to use the coordinate's application

$[Tx]_{\mathfrak{B}} = [c_1\,T(b_1) + \dots + c_nT\,(b_n)]_{\mathfrak{B}} = c_1\,[T(b_1)]_{\mathfrak{B}} + \dots + c_nT\,(b_n)]_{\mathfrak{B}}$

And writing this in matrix form

$$[Tx]_{\mathfrak{B}} = [[T(b_1)]_{\mathfrak{B}}|\ T(b_2)]_{\mathfrak{B}}|\ \dots\ |\ T(b_n)]_{\mathfrak{B}} \begin{bmatrix} c1 \\ c2 \\ \dots \\ cn \end{bmatrix} = [T]_{\mathfrak{B}}\ [x]_{\mathfrak{B}}$$

The columns of $P_{\mathfrak{B}}$ form a base of R^n, and the invertible matrix theorem (theorem 2.27 e. ´o h.) Implies that $P_{\mathfrak{B}}$ is invertible. We can say then that $P^{-1}{}_{\mathfrak{B}}$, which acts as

$$[x]_{\mathfrak{B}} = P^{-1}{}_{\mathfrak{B}}\ [x]_E$$

Is the matrix of the coordinate change from the canonical base to the base $\mathfrak{B}$.

But what if T is a linear application between two arbitrary vector spaces V and W? If the vector spaces are of finite dimension, we can use the coordinates, as we did to identify vectors with vectors of R^n, to identify the application with a matrix application from R^n to R^m.

The matrix of a linear application.

Let T: V→ W a linear applicationwhose domain is a vectors spaces V of dim n with whose codomino is a vectors spacesW of dim m. Using a base$\mathfrak{B}$ of V and a base C of W, we can associate to T a matrix application between R^n and R^m.

Indeed, given $x \in V$ we have that we can write any $x \in V$ in coordinates $[x]_{\mathfrak{B}}$ with respect to base $\mathfrak{B}$ and T(x) in coordinates $[T(x)]_C$ with respect to base C. Let $\mathfrak{B} = \{b_1,\dots, b_n\}$, and let be the coordinates of x

$$[x]_{\mathfrak{B}}=\begin{bmatrix} r1 \\ . \\ . \\ . \\ . \\ . \\ . \\ rn \end{bmatrix}$$

That is $x = r_1b_1 +\dots + r_nb_n$. So, since T is linear

$$y = T(x) = T(r_1b_1 + \_\ \_\ \_ + r_nb_n) = r_1T(b_1) + \_\ \_\ \_ + r_nT(b_n):$$

Writing this vector in coordinates with respect to C we have to

$[y]_C = [T(x)]_C = r_1[T(b_1)]_C + \_\ \_\ \_ + r_n[T(bn)]_C$

$$=[\ [T(b_1)]_C|[T(b_2)]_C|\_\ \_\ \_|\ [T(b_n)]_C] \begin{bmatrix} r1 \\ r2 \\ . \\ . \\ . \\ . \\ . \\ rn \end{bmatrix}$$

It should be noted, as it is easy to deduce from the previous example and from the associated matrix definition, that to fully determine a linear application T:V→ W is enough to give its value (in any base C of W) on the vectors of a base $\mathfrak{B}$ of V.

Example If V = W and the application is the identity T (x) = Id (x)=
Ix = x, the matrix
$[Id]_{C\leftarrow\mathfrak{B}} = [\ [Ib_1]_C|\ \dots|\ [Ib_n]_c\ ]\ ] = [\ [b_1]_C|\ \dots|Ib_n]_c\ ]\ ]$

Is the matrix of the base change of Theorem 3.61.

Matrix of a linear transformation of V. The particular case W = V oflinear transformations, that is, linear applications T: V→V that act on a space V, is very common. The normal thing is to use the same base $\mathfrak{B}$ of V to describe the images and the anti-images, that is, to use $[T]_{\mathfrak{B}\leftarrow\mathfrak{B}}$, matrix that is denoted by $[T]_{\mathfrak{B}}$ and is called $\mathfrak{B}$-matrix of T. With this, if y = T (x):

$[y]_{\mathfrak{B}} = [T(x)]_{\mathfrak{B}} = [T]_{\mathfrak{B}}\ [x]_{\mathfrak{B}}$paratodox $\in$V .

$[T]_{\mathfrak{B}}$ is also said to be the matrix of T at base$\mathfrak{B}$.

Example In Example 3.43 we coordinate the space $P_3$of polynomials of up to third degree, using the canonical base E= $\{1, t, t^2,t^3\}$. With this we saw that it is isomorphic to R^4: for a polynomialfrom P_3

$$p(t) = a_0 + a_1t + a_2t^2 + a_3t^3 \longleftrightarrow [p]_{\mathfrak{B}} = \begin{bmatrix} a0 \\ a1 \\ a2 \\ a3 \end{bmatrix}$$

The application D: P_3→ P_3, derived with respect to t is defined by

$p(t) = a_0 + a_1t + a_2t^2 + a3t^3 \rightarrow D(p(t)) = p'(t) = a_1 + 2a_2t + 3a_3t^2$

And we know, from what we know about the properties of the derivative,which is linear. Therefore, there will be an array that implements it as a matrix application of $R^4 \rightarrow R^4$. That matrix is calculated by puttingas columns the transformed vectors of the canonical base, incoordinates of the canonical base:

$D(1) = 0 \leftrightarrow [D(1)]_E \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $D(t) = 1 \leftrightarrow [D(t)]_E = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$D(t^2) = 2t \leftrightarrow [D(t^2)]_E = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$, $D(t^3) = 3t^2 \leftrightarrow [D(t^3)]_E = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$

So

$$[D]_E = [[D(1)]_C|[D(t)]_C|[D(t^2)]_C|[D(t^3)]_C] = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{3} \end{bmatrix}$$
$$\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}$$

It is easy to verify that

$$[D]_E \begin{bmatrix} a0 \\ a1 \\ a2 \\ a3 \end{bmatrix} == \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{3} \end{bmatrix} \begin{bmatrix} a0 \\ a1 \\ a2 \\ a3 \end{bmatrix} = \begin{bmatrix} a1 \\ 2a2 \\ 3a3 \\ 0 \end{bmatrix}$$
$$\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}$$

Which is equivalent to

$d/dt\ (a_0 + a_1t + a_2t^2 + a_3t^3) = a_1 + 2\ a_2t + 3\ a_3t^2$

Linear transformations of R^n. When in a vector space V we have a base $\mathfrak{B} = \{b_1,\ldots,b_n\}$, we have a way to describe its vectors as coordinate vectors, creating an isomorphism between V and R^n, and we have a way to describe their linear transformations as matrix transformations of R^n, using the $\mathfrak{B}$-matrix. When we despond of more than onebase, we have several ways to write coordinate vectors, and we have several ways to describe applications with matrices.

For example, consider R^n, with the canonical base E, and a diagonalizable matrix A of $n\times n$, with whose eigenvectors we can construct a base $\mathfrak{B}$ of R^n. There are two bases, we know the E-matrix of the linear application $x \rightarrow Ax$ (A itself), but that application would also have a $\mathfrak{B}$-matrix. That is, the application has matrix A in the canonical base, and a different matrix in base $\mathfrak{B}$.

Theorem. Suppose A is similar to a diagonal matrix D of $n\times n$, that is, $A = PDP^{-1}$. Theorem states that a column of P forms a base $\mathfrak{B}$ (of eigenvectors of A) of R^n. So D is a B-matrix of $x \rightarrow Ax$ on this basis:

$$\mathbf{[T]_E = A\ [T] = D}$$

Demonstration. If $\mathfrak{B} = \{b_1,.., b_n\}$ is the eigenvector base of A, so $Ab_i=$
$\boldsymbol{\lambda}_i b_i$, $i = 1,\ldots, n$ (there can be $\boldsymbol{\lambda}_i$ repeated if one has greater multiplicitythan 1). The matrix of $T(x) = Ax$ on that basis is

$[T]_{\mathfrak{B}} = [\ [T(b_1)]_{\mathfrak{B}}|\ \ldots|\ [T(b_n)]_{\mathfrak{B}}] = [\ [Ab_1]_{\mathfrak{B}}|\ \ldots|\ [Ab_n]_{\mathfrak{B}}\ ]$

$= [\ [\boldsymbol{\lambda}_1 b_1)]_{\mathfrak{B}}\ |\ \ldots|\ [\boldsymbol{\lambda}_n b_n)]_{\mathfrak{B}}] = [[\boldsymbol{\lambda}_1[b_1]_{\mathfrak{B}}\ |\ \ldots|\ [\boldsymbol{\lambda}_n[b_n]_{\mathfrak{B}}]$

$=[\boldsymbol{\lambda}_1\ e_1\ |\ \ldots|\ \boldsymbol{\lambda}_n\ e_n] = \begin{bmatrix} \lambda 1 & 0 & \cdots & 0 \\ 0 & \lambda 2 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda n \end{bmatrix} = D$

It is interesting to note that the matrix $P = [b_1|\ \ldots|b_n]$ is the matrix of change$P_{E\leftarrow\mathfrak{B}}$ base, as we will see in the next paragraph.

If $A = \begin{bmatrix} \mathbf{7} & \mathbf{2} \\ \mathbf{-4} & \mathbf{1} \end{bmatrix}$and $T(x) = Ax$, find a base of

the base will be the eigenvector base used when diagonal matrix. We already saw in that $A = PDP^{-1}$ with P havingas columns $b_1 = \begin{matrix} 1 \\ -1 \end{matrix}$,$b2 = \begin{matrix} 1 \\ -2 \end{matrix}$So statesthat $\mathfrak{B} = \{b_1, b_2\}$ is a base on which the application T hasmatrix

Similarity and change of coordinates of transformations. We have seen that if a matrix A is similar to another D, with $A = PDP^{-1}$, thenD is the matrix $[T]_{\mathfrak{B}}$ of T (x) = Ax at the base $\mathfrak{B}$given by the columns of P.Since A is the matrix $[T]_E$ of T in the canonical base, and $P = P_{E\leftarrow\mathfrak{B}}$is the matrix ofcoordinate change from $\mathfrak{B}$ to E, then

$A = PDP^{-1} \leftrightarrow [T]_E = P_{E\leftarrow\mathfrak{B}}[T]_{\mathfrak{B}} P_{\mathfrak{B}\leftarrow E}$

since $P^{-1} = P_{\mathfrak{B}\leftarrow E}$. This fact is general.

(Change of base of a matrix application). Let $\mathfrak{B}$ and C be twobases of R^n, and let T: $R^n \rightarrow R^n$ a linear application whose matrix in two basesis $[T]_{\mathfrak{B}}$ and $[T]_C$ respectively. So $[T]_C = P_{\mathfrak{B}\leftarrow C} [T]_{\mathfrak{B}} P_{\mathfrak{B}\leftarrow C}$, that is

$[T]_C = P\,[T]P^{-1} \leftrightarrow [T]_{\mathfrak{B}} = P^{-1}[T]_C\, P$

Being $P = [[b_1]_C| \dots |[b_n]_C]$the base change matrix $P_{c\leftarrow\mathfrak{B}}$.

A= $\begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}$, $b_1 = \frac{3}{2}$ and $b_2 = \frac{2}{1}$.

Find the $\mathfrak{B}$-matrix of x →Ax (matrix A in base) .

Solution: The base change matrix $P_{E\leftarrow\mathfrak{B}}$ is P =$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$, and inverse is $P^{-1} = P\mathfrak{B}\leftarrow E = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$, so that

$[T]_{\mathfrak{B}} = P_{\mathfrak{B}\leftarrow E} [T]_E P_{E\leftarrow\mathfrak{B}}$

$= P^{-1}AP = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$

One thing to note for later chapters: the first vectorb1$\frac{3}{2}$= is an eigenvector of A, with eigenvalue -2. The polynomialcharacteristic of A is $(\lambda + 2)^2$, so it only has an eigenvalue -2. Yeswe calculate their eigenvectors, we will discover that they are cb_1, the multiplesfrom b_1, and there is only one linearly independent one. Therefore, A **noit is diagonalizable**. Matrix $[T]_{\mathfrak{B}}$ is the best we can findsimilar to A: it is not diagonal, but at least it is triangular. I knowcalled Jordan's form of A.

6. Complex eigenvalues

The matrix A= $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$is a spin of + π/ 4. Your equationCharacteristic is

$$\lambda^2 + 1 = 0$$

For everything to make sense, we have to consider that the wholeof scalars is C instead of R, and that the vector vector spacecolumn is C^2 instead of R^2, in addition to that the matrices may bemade up of complex elements. With this expansion, you havefelt the complex diagonalization

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} j & -j \\ 1 & 1 \end{bmatrix}\begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix} 1/2j \begin{bmatrix} 1 & j \\ -1 & j0 \end{bmatrix}$

Diagonalizable the matrix A= $\begin{bmatrix} 7 & -8 \\ 5 & -5 \end{bmatrix}$ .The character equationsilica and the eigenvalues are

$\lambda^2\_2 + 5 = 0 = (\lambda \_ 1)^2 + 4^2) \rightarrow \lambda = 1 + 2j, 1 - 2j$

Note that eigenvalues are complexes conjugated to each other.This always happens if the matrix is 2 ×2 and real. The eigenvectors are also calculated at the same time, because it can be shown thatthey are also complex conjugates if A is real:

$A - (1 + 2j)I = \begin{bmatrix} 6-2j & -8 \\ 5 & -6+2j \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} 4 \\ 3-j \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 3+j \end{bmatrix}$

For all

$\begin{bmatrix} 7 & -8 \\ 5 & -5 \end{bmatrix} = P \begin{bmatrix} 1+2j & 0 \\ 0 & 1\_2j \end{bmatrix} P^{-1}$, con $P = \begin{bmatrix} 4 & 4 \\ 3-j & 3+j \end{bmatrix}$

The complex diagonal shape is not of much interest if we work with real matrices. There is a form of a real matrix that is not diagonal, but that would be very useful later, which is the one that adopts the matrix C of the following theorem.

Let A be a real matrix of 2× 2 with complex eigenvalue λ= a –bj (b≠0), and associated eigenvector v ∈ C^2. So

$A = P\,CP^{-1}$, where P = [Rev |Imv] and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

Note that in the case of the matrix in, thematrix A= $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,and the associated eigenvectors give rise to P = I, theidentity. The geometric interpretation of the transformation in R^2associated with A is that of a 90 degree rotation in a positive direction,being evident that this application does not have autovectors. Asare the other C transformations of R^2 that do not have eigenvectors ?Take example 6.40, from matrix A$\begin{bmatrix} 7 & -8 \\ 5 & -5 \end{bmatrix}$:= eigenvaluesare 1 - 2j and 1 + 2j, and the eigenvector corresponding to 1 - 2j was$\begin{bmatrix} 4 \\ 3+j \end{bmatrix}$then P= $\begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix}$, C= $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$and the factorization is

$A = PCP^{-1} \rightarrow \begin{bmatrix} 7 & -8 \\ 5 & -5 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} 1/4 \begin{bmatrix} 1 & 0 \\ -3 & 4 \end{bmatrix}$

The "geometric interpretation of the transformation associated with A= $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$it is a rotation of angle arctanb / a and also a dilation ofmagnitude$\sqrt{a^2 + b^2}$ (see formula (5.2))

7. Summary

An eigenvector (orauto vector) of a matrix A of n×nis a vector x ∈ R^n, other than, such thatfor true scale**λ**∈ RAx = **λ**x.Like the scalar **λ**is call a eigenvalues (or eigenvalue) of A, that is,**λ** it is a eigenvalues of A if there is the not trivial solve of Ax = **λ**x, with x is call a eigenvectors associate anda eigenvalue**λ**.

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