Generalised k- Jacobsthal 2^m Ions (For Fixed m) Quarternions, Sedenions

G.Srividhya[a], and E.Kavitha rani[b]
[a]
Assistant Professor, PG & Research Department of Mathematics,
Government Arts College.Trichirappalli-22.
[b]Guest lecturer. PG & Research Department of Mathematics, Government Arts College. Trichirappalli-22.

Article History: Received: 10 January 2021; Revised: 12 February 2021; Accepted: 27 March 2021; Published online: 20 April 2021

Abstract: In this paper we deliberate about Generalised k- Jacobsthal Quaternions, Octonions, and Sedenions. We discuss Binet formula, Generating function, Catalan Identity, Cassini Identity, D'Ocagne's Identity of them. From that we extent the same results for k- Jacobsthal, Generalised k- Jacobsthal.

1. Introduction

Basic Definitions

Generalized k-Jacobsthal Number

Let k be any positive real number. $f(k), g(k)$ are scalar valued polynomials for $f^2(k) + 8g(k) > 0$, for $n \in N$ generalized k-Jacobsthal sequence $J_{k,n}$ is defined as

$$J_{k,n} = f(k)J_{k,n-1} + 2g(k)J_{k,n-2}, J_{k,0} = a, J_{k,b} = b, n \geq 2 \quad (1)$$

Binet form of Generalised k-Jacobsthal Number

$$J_{k,n} = \frac{X\alpha^n - Y\beta^n}{\alpha - \beta} \quad (2)$$

Where $X = b - a\beta, Y = b - a\alpha$

$$\alpha = \frac{f(k) + \sqrt{f^2(k) + 8g(k)}}{2}, \beta = \frac{f(k) - \sqrt{f^2(k) + 8g(k)}}{2}$$

Here α, β are the root of the characteristic equation $x^2 - f(k)x - 2g(k) = 0$.

The Cayley-Dickson algebra are sequence $A_0, A_1, \ldots$ of non-associative R-algebra with involution. Let us defining A_0 be R. Given A_{m-1} is defined additively to be $A_{m-1} * A_{m-1}$ conjugation in A_m is defined by

$$\left(\overline{a, b}\right) = (\bar{a}, -b)$$

Multiplication is defined by $(a, b).(c, d) = \left(ac - \bar{d}b, da + b\bar{c}\right)$

Addition is defined by component wise as

$$(a, b) + (c, d) = (a + c, b + d)$$

A_m has dimension $N = 2^m$ as an R-vector space. If $\|x\| = \sqrt{Re(x\bar{x})}$ for $x \in A_m$ then

$$x\bar{x} = \bar{x}x = \|x\|^2$$

for specific $m, 2^m$ is tabulated below

m	2	3	4	…
2^m	Quarternions	Octonions	Sedenions	…

for a fixed m. Suppose $B_N = e_i \in A_m, i = 0,1,2, \ldots N - 1$ is the basis for A_m where $N = 2^m$ is the dimension of A_m, e_0 is the identity (or unit) and $e_1, e_2, \ldots, e_{N-1}$ are called imaginaries. Then 2^m ions $s \in A_m$ taken as

$$s = \sum_{i=0}^{N-1} a_i e_i = a_0 + \sum_{i=1}^{N-1} a_i e_i$$

where $a_0, a_1, \ldots, a_{N-1}$ are real numbers. Here $a_0$is called the real part of s and $\sum_{i=1}^{N-1} a_i e_i$ is called imaginary part.

Generalised k-Jacobsthal 2^m ions

Generalised k-Jacobsthal 2^m ions sequence $\{\hat{G}J_{k,n}\}_{n\geq 0}$ is defined by

$$\hat{G}J_{k,n} = \sum_{s=0}^{N-1} J_{k,n+s} e_s \quad (3)$$

Let us define Generalised k-Jacobsthal 2^m ions such as Quaternions, Octonions, and Sedenions as follows

(a) Put $N = 4$ in (3) we get Generalised k-Jacobsthal Quarternions $\hat{G}Q_{k,n}$

$$\hat{G}Q_{k,n} = J_{k,n} + J_{k,n+1}e_1 + J_{k,n+2}e_2 + J_{k,n+3}e_3$$
$$= \sum_{s=0}^{3} J_{k,n+s} e_s$$

(b) By substituting $N = 8$ in (3) we get Generalised k-Jacobsthal Octonions $\hat{G}Q_{k,n}$

$$\hat{G}Q_{k,n} = \sum_{s=0}^{7} J_{k,n+s} e_s$$

(c) By substituting $N = 16$ in (3) we get Generalised k-Jacobsthal Octonions $\hat{G}S_{k,n}$

$$\hat{G}S_n = \sum_{s=0}^{15} J_{k,n+s} e_s$$

Where $J_{k,n}$ is n^{th} generalized k-Jacobsthal number.

From the equation (1),(2) we have the following recurrence relation

$$\hat{G}J_{k,n} = f(k)\hat{G}J_{k,n-1} + 2g(k)\hat{G}J_{k,n-2}, \hat{G}J_{k,0} = a, \hat{G}J_{k,1} = b \quad n \geq 2 \tag{4}$$

For specific values of $a, b, f(k), g(k)$ we present some specific sequences

S.No	$(a, b, f(k), g(k))$	Name of the sequences
1	$(0,1,1,1)$	Jacobsthal
2	$(0,1,k,1)$	k-Jacobsthal
3	$(0,1,k,-1)$	Derived k-Jacobsthal

Let $2^m = N$, we fix the following Notations

$$\bar{\alpha} = \sum_{s=0}^{N-1} \alpha^s e_s \qquad \bar{\beta} = \sum_{s=0}^{N-1} \beta^s e_s$$

Theorem 1

Binet form of Generalized k-Jacobsthal 2^m ions

$$\hat{G}J_{k,n} = \frac{X\bar{\alpha}\alpha^n - Y\bar{\beta}\beta^n}{\alpha - \beta} \tag{5}$$

where

$$\bar{\alpha} = \sum_{s=0}^{N-1} \alpha^s e_s, \qquad \bar{\beta} = \sum_{s=0}^{N-1} \beta^s e_s$$

$$X = b - a\beta, \qquad Y = b - a\alpha$$

$$\alpha = \frac{f(k) + \sqrt{f^2(k) + 8g(k)}}{2}, \quad \beta = \frac{f(k) - \sqrt{f^2(k) + 8g(k)}}{2}$$

Proof:

Using (2), (3)

$$\hat{G}J_{k,n} = \sum_{s=0}^{N-1} J_{k,n+s}\, e_s$$

$$= \left(\frac{X\alpha^n - Y\beta^n}{\alpha - \beta}\right)e_0 + \left(\frac{X\alpha^{n+1} - Y\beta^{n+1}}{\alpha - \beta}\right)e_1 + \left(\frac{X\alpha^{n+2} - Y\beta^{n+2}}{\alpha - \beta}\right)e_2 + \cdots + \left(\frac{X\alpha^{n+N-1} - Y\beta^{n+N-1}}{\alpha - \beta}\right)e_{N-1}$$

Doing simplification we get

$$\hat{G}J_{k,n} = \frac{X\bar{\alpha}\alpha^n - Y\bar{\beta}\beta^n}{\alpha - \beta}$$

Proposition 1.1

Binet form for Generalised k-Jacobsthal Quaternions

From (5)

$$\hat{G}Q_{k,n} = \frac{X\bar{\alpha}\alpha^n - Y\bar{\beta}\beta^n}{\alpha - \beta} \tag{6}$$

Where

$$\bar{\alpha} = \sum_{s=0}^{3} \alpha^s e_s, \qquad \bar{\beta} = \sum_{s=0}^{3} \beta^s e_s$$

$$X = b - a\beta, \qquad Y = b - a\alpha$$

$$\alpha = \frac{f(k) + \sqrt{f^2(k) + 8g(k)}}{2}, \quad \beta = \frac{f(k) - \sqrt{f^2(k) + 8g(k)}}{2}$$

Corollary 1.1.1

Binet form for k-Jacobsthal Quarternions

Let from (6), $f(k) = k, g(k) = 1, a = 0, b = 1$ then $X = 1, Y = 1$

$$\hat{Q}_{k,n} = \frac{\bar{\alpha}\alpha^n - \bar{\beta}\beta^n}{\alpha - \beta}$$

where

$$\bar{\alpha} = \sum_{s=0}^{3} \alpha^s e_s, \qquad \bar{\beta} = \sum_{s=0}^{3} \beta^s e_s$$

$$\alpha = \frac{k+\sqrt{k^2+8}}{2}, \quad \beta = \frac{k-\sqrt{k^2+8}}{2}$$

Corollary 1.1.2

Binet form for Derived k-Jacobsthal Quarternions

Let from (6), $f(k) = k, g(k) = -1, a = 0, b = 1$ then $X = 1, Y = 1$

$$\widehat{D}Q_{k,n} = \frac{\bar{\alpha}\alpha^n - \bar{\beta}\beta^n}{\alpha - \beta}$$

where

$$\bar{\alpha} = \sum_{s=0}^{3} \alpha^s e_s, \qquad \bar{\beta} = \sum_{s=0}^{3} \beta^s e_s$$

$$\alpha = \frac{k+\sqrt{k^2-8}}{2}, \quad \beta = \frac{k-\sqrt{k^2-8}}{2}$$

Proposition 1.2

Binet form for Generalised k-Jacobsthal Octonions

From (5)

$$\widehat{G}O_{k,n} = \frac{X\bar{\alpha}\alpha^n - Y\bar{\beta}\beta^n}{\alpha - \beta} \tag{7}$$

where

$$\bar{\alpha} = \sum_{s=0}^{7} \alpha^s e_s, \qquad \bar{\beta} = \sum_{s=0}^{7} \beta^s e_s$$

The values of X, Y, α, β are same as in Proposition 1.1

Corollary 1.2.1

Binet form for k-Jacobsthal Octonions

Let from (7), $f(k) = k, g(k) = 1, a = 0, b = 1$ then $X = 1, Y = 1$

$$\widehat{O}_{k,n} = \frac{\bar{\alpha}\alpha^n - \bar{\beta}\beta^n}{\alpha - \beta}$$

where

$$\bar{\alpha} = \sum_{s=0}^{7} \alpha^s e_s, \qquad \bar{\beta} = \sum_{s=0}^{7} \beta^s e_s$$

$$\alpha = \frac{k+\sqrt{k^2+8}}{2}, \quad \beta = \frac{k-\sqrt{k^2+8}}{2}$$

Corollary 1.2.2

Binet form for Derived k-Jacobsthal Octonions

Let from (7), $f(k) = k, g(k) = -1, a = 0, b = 1$ then $X = 1, Y = 1$

$$\widehat{D}O_{k,n} = \frac{\bar{\alpha}\alpha^n - \bar{\beta}\beta^n}{\alpha - \beta}$$

where

$$\bar{\alpha} = \sum_{s=0}^{7} \alpha^s e_s, \qquad \bar{\beta} = \sum_{s=0}^{7} \beta^s e_s$$

$$\alpha = \frac{k+\sqrt{k^2-8}}{2}, \quad \beta = \frac{k-\sqrt{k^2-8}}{2}$$

Proposition 1.3

Binet form for Generalised k-Jacobsthal Sedenions

From (5)

Let $N = 16$

$$\widehat{G}S_{k,n} = \frac{X\bar{\alpha}\alpha^n - Y\bar{\beta}\beta^n}{\alpha - \beta} \tag{8}$$

where

$$\bar{\alpha} = \sum_{s=0}^{15} \alpha^s e_s, \qquad \bar{\beta} = \sum_{s=0}^{15} \beta^s e_s$$

The values of X, Y, α, β are same as in Proposition 1.1

Corollary 1.3.1

Binet form for $\boldsymbol{k}$-Jacobsthal Sedenions

Let from (8), $f(k) = k, g(k) = 1, a = 0, b = 1$ then $X = 1, Y = 1$

$$\hat{S}_{k,n} = \frac{\bar{\alpha}\alpha^n - \bar{\beta}\beta^n}{\alpha - \beta}$$

where

$$\bar{\alpha} = \sum_{s=0}^{15} \alpha^s e_s, \qquad \bar{\beta} = \sum_{s=0}^{15} \beta^s e_s$$

$$\alpha = \frac{k + \sqrt{k^2 + 8}}{2}, \quad \beta = \frac{k - \sqrt{k^2 + 8}}{2}$$

Corollary 1.3.2

Binet form for Derived $\boldsymbol{k}$-Jacobsthal Sedenions

Let from (8), $f(k) = k, g(k) = -1, a = 0, b = 1$ then $X = 1, Y = 1$

$$\widehat{D}S_{k,n} = \frac{\bar{\alpha}\alpha^n - \bar{\beta}\beta^n}{\alpha - \beta}$$

where

$$\bar{\alpha} = \sum_{s=0}^{15} \alpha^s e_s, \qquad \bar{\beta} = \sum_{s=0}^{15} \beta^s e_s$$

$$\alpha = \frac{k + \sqrt{k^2 - 8}}{2}, \quad \beta = \frac{k - \sqrt{k^2 - 8}}{2}$$

Theorem 2

Generating function for Generalized $\boldsymbol{k}$ − Jacobsthal $\boldsymbol{2^m}$ ions

$$G(t) = \frac{\hat{G}J_{k,0} + \left(\hat{G}J_{k,1} - f(k)\hat{G}J_{k,0}\right)t}{1 - f(k)t - 2g(k)t^2}$$

Proof

Let $G(t) = \sum_{n=0}^{\infty} \hat{G}J_{k,n}t^n$ be the generating function of k − Jacobsthal 2^m ions, then

$(1 - f(k)t - 2g(k)t^2) = \widehat{(G}J_{k,0} + \hat{G}J_{k,1}t) - f(t)\hat{G}J_{k,0}t + \sum_{n=0}^{\infty}(\hat{G}J_{k,n} - f(t)\hat{G}J_{k,n-1} - 2g(t)\hat{G}J_{k,n-2})t^n$

Doing simple calculation we get

$$G(t) = \frac{\hat{G}J_{k,0} + \left(\hat{G}J_{k,1} - f(k)\hat{G}J_{k,0}\right)t}{1 - f(k)t - 2g(k)t^2}$$

Examples

1. Generating function for k − Jacobsthal Quaternions

$$f(k) = k, g(k) = 1, \hat{G}J_{k,0} = 0, \hat{G}J_{k,1} = 1$$

$$G(t) = \frac{t}{1 - kt - 2t^2}$$

2. Generating function for Derived k − Jacobsthal Quaternions

$$f(k) = k, g(k) = -1, \hat{G}J_{k,0} = 0, \hat{G}J_{k,1} = 1$$

$$G(t) = \frac{t}{1 - kt + 2t^2}$$

Theorem 3

Catalan's identity for Generalized $\boldsymbol{k}$ − Jacobsthal $\boldsymbol{2^m}$ ions

For any positive integer $p, q, p > q$

$$\hat{G}J_{k,p-q}.\hat{G}J_{k,p+q} - \hat{G}J_{k,p}^2 = \frac{XY(\alpha\beta)^p - (\beta^{-q} - \alpha^{-q})\left(\beta^q\bar{\alpha}\bar{\beta} - \alpha^q\bar{\beta}\bar{\alpha}\right)}{(\alpha - \beta)^2} \tag{9}$$

Where $X, Y, \alpha, \beta, \overline{\alpha, \beta}$ are same in equation (5)

Proof

Using Binet form

$$\hat{G}J_{k,p-q}.\hat{G}J_{k,p+q} - \hat{G}J_{k,p}^2 = \frac{X\bar{\alpha}\alpha^{p-q} - Y\bar{\beta}\beta^{p-q}}{\alpha - \beta}\frac{X\bar{\alpha}\alpha^{p+q} - Y\bar{\beta}\beta^{p+q}}{\alpha - \beta} - \left(\frac{X\bar{\alpha}\alpha^p - Y\bar{\beta}\beta^p}{\alpha - \beta}\right)^2$$

Doing simple mathematical simplification we get the result.

Proposition 3.1

Catalan Identity for Generalized $\boldsymbol{k}$ − Jacobsthal Quaternions:

Same result of Theorem 3.where

$$\bar{\alpha} = \sum_{s=0}^{3} \alpha^s e_s, \quad \bar{\beta} = \sum_{s=0}^{3} \beta^s e_s$$

Corollary 3.1.1
Catalan Identity for $k-$ Jacobsthal Quarternions:
For any positive integer p, q such that $p > q$

$$\hat{G}J_{k,p-q}.\hat{G}J_{k,p+q} - \hat{G}J_{k,p}^2 = -\bar{\alpha}\bar{\beta}(-2)^{p-q}\hat{G}J_{k,p}^2 \qquad (10)$$

Proof:
Let $f(k) = k, g(k) = 1, \hat{G}J_{k,0} = a = 0, \hat{G}J_{k,1} = b = 1$ then $X = 1, Y = 1$, $\bar{\alpha} = \sum_{s=0}^{3} \alpha^s e_s$, $\bar{\beta} = \sum_{s=0}^{3} \beta^s e_s$, $\alpha = \frac{k+\sqrt{k^2+8}}{2}$, $\beta = \frac{k-\sqrt{k^2+8}}{2}$ using all above values in equation (10) we get the result

Corollary 3.1.2
Catalan Identity for Derived k-Jacobsthal Quarternions:
$f(k) = k, g(k) = -1, \hat{G}J_{k,0} = a = 0, \hat{G}J_{k,1} = b = 1$, then $X = 1, Y = 1$,

$$\alpha = \frac{k+\sqrt{k^2-8}}{2}, \beta = \frac{k-\sqrt{k^2-8}}{2}, \bar{\alpha} = \sum_{s=1}^{3} \alpha^s e_s, \bar{\beta} = \sum_{s=1}^{3} \beta^s e_s$$

using all above values in equation (10) we get

$$\hat{G}J_{k,p-q}\hat{G}J_{k,p+q} - \hat{G}J_{k,p}^2 = -(\bar{\alpha}\,\bar{\beta}\,)2^{p-q}\hat{G}J_{k,q}^2$$

Proposition 3.2
Catalan Identity for Generalised k-Jacobsthal Octonions: Same result of Theorem 3. Where $\bar{\alpha} = \sum_{s=1}^{7} \alpha^s e_s, \bar{\beta} = \sum_{s=1}^{7} \beta^s e_s$
Note
For k-Jacobsthal Octonions, Derived k-Jacobsthal Octonions: Same result of corollary 3.1.1, corollary 3.1.2 where $\bar{\alpha} = \sum_{s=1}^{7} \alpha^s e_s, \bar{\beta} = \sum_{s=1}^{7} \beta^s e_s$
Proposition 3.3
Catalan Identity for Generalised k-Jacobsthal Sedenions: Some result of Theorem 3. Where $\bar{\alpha} = \sum_{s=0}^{15} \alpha^s e_s, \bar{\beta} = \sum_{s=0}^{15} \beta^s e_s$
Note
For k-Jacobsthal Sedenions, Derived k-Jacobsthal Sedenions: Same result of corollary 3.1.1, corollary 3.1.2 where $\bar{\alpha} = \sum_{s=0}^{15} \alpha^s e_s, \bar{\beta} = \sum_{s=0}^{15} \beta^s e_s$
Theorem 4
Cassini Identity Taking $q = 1$ in Catalan's Identity we get Cassini Identity of Generalised k-Jacobsthal 2^m −ions.

$$\hat{G}J_{k,p-1}\hat{G}J_{k,p+1} - \hat{G}J_{k,p}^2 = \frac{XY(\alpha\beta)^{p-1}(\beta\bar{\alpha}\bar{\beta} - \alpha\bar{\beta}\bar{\alpha})}{\alpha-\beta} \qquad (11)$$

Where $X, Y, \alpha, \beta, \bar{\alpha}, \bar{\beta}$ are same in equation (5).
Note
Cassini identity for Generalised k-Jacobsthal Quarterions, Octonions, Sedenions having same result of Theorem 4, where the values of $\bar{\alpha}, \bar{\beta}$ are to be choosen corresponding

$$\bar{\alpha} = \sum_{s=1}^{3} \alpha^s e_s, \bar{\beta} = \sum_{s=1}^{3} \beta^s e_s$$
$$\bar{\alpha} = \sum_{s=1}^{7} \alpha^s e_s, \bar{\beta} = \sum_{s=1}^{7} \beta^s e_s$$
$$\bar{\alpha} = \sum_{s=1}^{15} \alpha^s e_s, \bar{\beta} = \sum_{s=1}^{15} \beta^s e_s$$

Theorem 5
D'ocagne's Identity for Generalised k-Jacobsthal 2^m ions
For any integer p, q

$$\hat{G}J_{k,p}\hat{G}J_{k,q+1} - \hat{G}J_{k,p+1}\hat{G}J_{k,q} = \frac{XY(\alpha^q\beta^p - \alpha^p\beta^q)(\beta\bar{\alpha}\bar{\beta} - \alpha\bar{\beta}\bar{\alpha})}{(\alpha-\beta)^2} \qquad (12)$$

Proof
Using Binet formula and simple mathematical simplification we can prove this result.
Proposition 5.1
D'ocagene's Identity for Generalised k-Jacobsthal Quarternions: Some result of Theorem 5. where

$$\bar{\alpha} = \sum_{s=1}^{3} \alpha^s e_s, \bar{\beta} = \sum_{s=1}^{3} \beta^s e_s$$

Corollary 5.1.1

D'ocagene's Identity for $\boldsymbol{k}$-Jacobsthal Quarternions: If $p > q$ then

$$\hat{G}J_{k,p}\hat{G}J_{k,q+1} - \hat{G}J_{k,p+1}\hat{G}J_{k,q} = (\bar{\alpha}\bar{\beta})(-2)^q\hat{G}J_{k,p-q}^2$$

Proof

Let us to be $f(k) = k, g(k) = 1, \hat{G}J_{k,0} = a = 0, \hat{G}J_{k,1} = b = 1,$ then

$$X = 1, Y = 1, \bar{\alpha} = \sum_{s=1}^{3} \alpha^s e_s, \bar{\beta} = \sum_{s=1}^{3} \beta^s e_s$$

$$\alpha = \frac{k + \sqrt{k^2 + 8}}{2}, \beta = \frac{k - \sqrt{k^2 + 8}}{2}$$

Using all above in Theorem 5 we get the result.

Corollary 5.1.2

D'ocagene's Identity for Derived $\boldsymbol{k}$-Jacobsthal Quarternions: If $p > q$ then

$$\hat{G}J_{k,p}\hat{G}J_{k,q+1} - \hat{G}J_{k,p+1}\hat{G}J_{k,q} = (\bar{\alpha}\bar{\beta})2^q\hat{G}J_{k,p-q}^2$$

Proof

Taking all the values as in corollary 5.1.1 except $g(k) = -1$,

$\alpha = \frac{k+\sqrt{k^2-8}}{2}, \beta = \frac{k-\sqrt{k^2-8}}{2}$ using all above in Theorem 5 we get the result.

Note

D'ocagene's Identity for Generalized k-Jacobsthal Octonions, Sedenions can be derived in the same way as in Proposition 5.1.

Conclusion

In this paper we discussed Generalized k-Jacobsthal Quartertions, Octonions, Sedenions. We explain Binet form, Generating function, Catalan Identity, D'ocagene's Identity of Generalized k-Jacobsthal 2^m ions. From that deduce the same result for k-Jacobsthal. In future we may also produce an extension of the above result for Generalised k-Jacobsthal Lucas, k Pell Lucas.

References

1. "A not on generalized k-horadam sequence" Yasin Yazlik, Necati Taskara, Elsevier, October 2011.
2. "Horadam 2^k-ions Melih Gocen and Yuksel soykan Preprints(www.Preprints.org)
3. "Generalized k-Jacobsthal sequence" S. Uygan and A. Tumbas. Asian Journal of Mathematics and Physics Vol2. Issue 2, 2018 45-50.