The Beckman-Quarles Theorem For Rational Spaces: Mappings Of Q^5 To Q^5 That Preserve Distance 1

By: Wafiq Hibi
Wafiq. hibi@gmail.com
The college of sakhnin - math department

Abstract: Let R^d and Q^d denote the real and the rational d-dimensional space, respectively, equipped with the usual Euclidean metric. For a real number $\rho > 0$, a mapping $f: A \longrightarrow X,$ where X is either R^d or Q^d and $A \subseteq X$, is called ρ-distance preserving $\|x-y\| = \rho$ implies $\|f(x)-f(y)\| = \rho$, for all x,y in A.
Let G(Q^d,a) denote the graph that has Q^d as its set of vertices, and where two vertices x and y are connected by edge if and only if $\|x-y\| = a$. Thus, G(Q^d,1) is the unit distance graph. Let ω(G) denote the clique number of the graph G and let ω(d) denote ω(G(Q^d, 1)).
The Beckman-Quarles theorem [1] states that every unit- distance-preserving mapping from R^d into R^d is an isometry, provided $d \geq 2$.
The rational analogues of Beckman- Quarles theorem means that, for certain dimensions d, every unit- distance preserving mapping from Q^d into Q^d is an isometry.
A few papers [2, 3, 4, 5, 6, 8,9,10 and 11] were written about rational analogues of this theorem, i.e, treating, for some values of d, the property "Every unit- distance preserving mapping $f: Q^d \longrightarrow Q^d$ is an isometry".
The purpose of this thesis is to prove the following Theorem.
Theorem 1:
Every unit- distance preserving mapping f: $Q^5 \rightarrow Q^5$ is an isometry; moreover, *dim(aff(f(L[5])))=5*.

1.1 Introduction:

Let R^d and Q^d denote the real and the rational d-dimensional space, respectively.
Let $\rho > 0$ be a real number, a mapping : $R^d \longrightarrow Q^d$, is called ρ- distance preserving if $\|x-y\| = \rho$ implies $\|f(x)-f(y)\| = \rho$.

The Beckman-Quarles theorem [1] states that every unit- distance-preserving mapping from R^d into R^d is an isometry, provided$d \geq 2$.
A few papers [4, 5, 6, 8,9,10 and 11] were written about the rational analogues of this theorem, i.e, treating, for some values of d, the property "every unit- distance preserving mapping $f: Q^d \longrightarrow Q^d$ is isometry".

We shall survey the results from the papers [2, 3, 4, 5, 6, 8,9,10 and 11] concerning the rational analogues of the Backman-Quarles theorem, and we will extend them to all the remaining dimensions , $d \geq 5$.

History of the rational analogues of the Backman-Quarles theorem:
We shall survey the results from papers [2, 3, 4, 5, 6, 8,9,10 and 11] concerning the rational analogues of the Backman-Quarles theorem.

1. A mapping of the rational space Q^d into itself, for d=2, 3 or 4, which preserves all unit- distance is not necessarily an isometry; this is true by W.Bens [2, 3] and H.Lenz [6].

2. W.Bens [2, 3] had shown the every mapping $f: Q^d \longrightarrow Q^d$ that preserves the distances 1 and 2 is an isometry, provided $d \geq 5$.

3. Tyszka [8] proved that every unit- distance preserving mapping $f: Q^8 \longrightarrow Q^8$ is an isometry; moreover, he showed that for every two points x and y in Q^8 there exists a finite set S_{xy} in Q^8 containing x and y such that every unit- distance preserving mapping $f: S_{xy} \longrightarrow Q^8$ preserves the distance between x and y. This is a kind of compactness argument, that shows that for every two points x and y in Q^d there exists a finite set S_{xy}, that contains x and y ("a neighborhood of x and y") for which already every unit- distance preserving mapping from this neighborhood of x and y to Q^d must preserve the distance from x to y. This implies that every unit preserving mapping from Q^d to Q^d must preserve the distance between every two points of Q^d.

4. J.Zaks [8, 9] proved that the rational analogues hold in all the even dimensions d of the form $d = 4k(k+1)$, for $k \geq 1$, and they hold for all the odd dimensions d of the form $d = 2n^2 - 1 = m^2$. For integers $n, m \geq 2$, (in [9]), or $d = 2n^2 - 1$, $n \geq 3$ (in [10]).

5. R.Connelly and J.Zaks [5] showed that the rational analogues hold for all even dimensions $d, d \geq 6$.

We wish to remark that during the preparation of this thesis, it was pointed out to us that an important argument, in the proof of the even dimensions $d, d \geq 6$, is missing. Here we propose a valid proof for all the cases of $d, d \geq 5$.

6. J.Zaks [11] had shown that every mapping $f: Q^d \longrightarrow Q^d$ that preserves the distances 1 and $\sqrt{2}$ is an isometry, provided $d \geq 5$.

New results:

Denote by $L[d]$ the set of $4 \cdot \binom{d}{2}$ Points in Q^d in which precisely two non-zero coordinates are equal to 1/2 or -1/2.

A "quadruple" in $L[d]$ means here a set $L_{ij}[d]$, $i \neq j \in I = \{1, 2, \ldots, d\}$; contains four j points of $L[d]$ in which the non- zero coordinates are in some fixed two coordinates i and j; i.e.

$$L_{ij}[d] = (0, \ldots 0, \underset{i}{\pm \tfrac{1}{2}}, 0 \ldots 0, \underset{j}{\pm \tfrac{1}{2}}, 0, \ldots 0)$$

Our main results are the following:

Theorem 1:
Every unit- distance preserving mapping $f: Q^5 \longrightarrow Q^5$ is an isometry; moreover, dim $(aff(f(L[5]))) = 5$.

Hibi prove the following lemma:
If x and y are two points in $Q^d, d \geq 5$, so that:

$$\sqrt{2 + \frac{2}{m-1}} - 1 \leq \|x - y\| \leq \sqrt{2 + \frac{2}{m-1}} + 1$$

where $\omega(d) = m$, then there exists a finite set S(x,y), contains x and y such that $f(x) \neq f(y)$ holds for every unit- distance preserving mapping $f: S(x,y) \rightarrow Q^d$.

Corollary:
If x and y are two points in $Q^d, d \geq 5$, such that $\|x - y\| = \sqrt{2}$, then every unit- distance preserving mapping $f: Q^d \rightarrow Q^d$ satisfies $f(x) \neq f(y)$.

Mappings of Q^5 to Q^5 that preserve distance 1

The purpose of this section is to prove the following Theorem.

Theorem 1:
Every unit- distance preserving mapping $f: Q^5 \rightarrow Q^5$ is an isometry; moreover, $dim(aff(f(L[5]))) = 5$.
To prove Theorem 1, we prove first the following Theorem.

Theorem 1*:
If Z, W are two points in Q^5, for which $\|Z - W\| = \sqrt{2}$, then there exists a finite set M_5, containing Z and W, such that for every unit- distance preserving mapping $f: M_5 \rightarrow Q^5$, the following equality holds:
$\|f(Z) - f(W)\| = \|Z - W\|$

Proof of Theorem 1*:
Let Z, W are any two points in Q^5, for which $\|Z - W\| = \sqrt{2}$.
Denote by $L[5]$ the set of $4 \cdot \binom{5}{2} = 40$ points in Q^d in which precisely two coordinates are non- zero and are equal to 1/2 or -1/2 .
A "quadruple" in $L[5]$ means a set $L_{ij}[5], i \neq j \epsilon I = \{1, 2, 3, 4, 5\}$, containing four points of $L[5]$ in which the non-zero coordinates are in some fixed two, the i-th and the j-th coordinates; i.e.

$L_{ij}[5] = \{\left(0, \pm\frac{1}{2}, 0, \pm\frac{1}{2}, 0\right)\}$ *1 i . j 5*

If ρ is a distance between any two points of the set *L[5]* then $\rho \epsilon \{\sqrt{0.5}, 1, \sqrt{1.5}, \sqrt{2}\}$.

Fix a quadruple $L_{ij}[5]$ let *x, y* two points in $L_{ij}[5]$ such that $\|x\text{-}y\| = \sqrt{2}$.

By Lemma 1 and based on $\|Z\text{-}W\| = \|x\text{-}y\|$, there exists a rational isometry h: $Q^5 \rightarrow Q^5$ for which *h(x) =:Z=x** and *h(y)=W:=y** ; denote *h(l)=l** for all lϵ *L[5]*.

Let *L*[5] = {l* = h(l)* for all lϵ *L[5]}*; it is clear that *Z, W* ϵ *L*[5]*, and to simplify terminology we will denote *L*[5]* = $\{l^*_i\}$ when $i \epsilon \{1, 2, \ldots, 40\}$.

Define the set M_5 by: $M_5 = \cup \{ S(l^*_i, l^*_j) \cup S(l^*_n, l^*_m) \cup S(l^*_s, l^*_t) \}$;

for all *i, j, n, m, s ,t*ϵ {1, 2, …, 40} when $\|l^*_i - l^*_j\| = \sqrt{0.5}$,

$\|l^*_n - l^*_n\| = \sqrt{1.5}$ and $\|l^*_s - l^*_t\| = \sqrt{2}$; where the sets *S* are given by Lemma 4.

Let *f, f*: $M_5 \rightarrow Q^5$ be any unit- distance preserving mapping.

Claim 1:

If *x* and *y* are two points in *L*[5]* for which $\|x\text{-}y\| = 1, \sqrt{2}$ then *f(x) ≠ f(y).*

Proof of Claim 1:

Clearly, if $\|x\text{-}y\|=1$, then $\|f(x) - f(y)\|=1$, hence *f(x) ≠ f(y).*

The distance $\sqrt{2}$ is between $\sqrt{2+\frac{2}{m-1}} - 1$ and $\sqrt{2+\frac{2}{m-1}} + 1$.

Where *m=ω(d)*=4 for *d*=5.

Therefore, if $\|x\text{-}y\| = \sqrt{2}$, then there exist an *i* and *j*, $1 \leq i \neq j \leq 40$, such that $x = l^*_i$, $y = l^*_j$ and $\|l^*_i - l^*_j\| = \sqrt{2}$. (l^*_i and l^*_j on the same quadruple).

By Lemma 4, applied to l^*_i *and* l^*_j, there exists a set S(l^*_i, l^*_j), that contains l^*_i *and* l^*_j, for which every unit-distance preserving mapping g: S(l^*_i, l^*_j)$\rightarrow Q^5$ satisfies

g(l^*_i)≠ g(, l^*_j).

In particular this holds for the mapping *g= f* / S(l^*_i, l^*_j), therefore f(l^*_i)≠ f(, l^*_j).

Claim 2:

The mapping f preserves all the distances$\sqrt{2}$. In particular $\|f(Z)\text{-}f(W)\| = \sqrt{2}$.

Proof of Claim 2:

Consider the graph P of unit distances among the points of *L*[5]*; it is isomorphic to the famous Petersen's graph, by substituting a 4-cycle for each vertex of P.

(See figure 4).

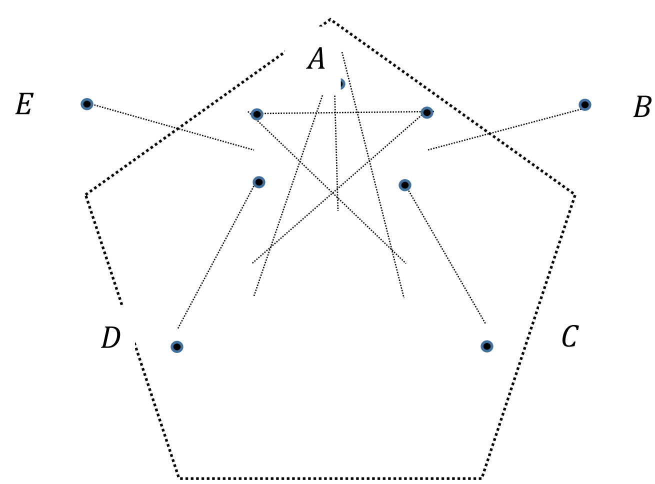

Figure 4

We prove that the affine dimension of the *f*- image of each quadruple, i.e., the image of the four points that correspond to one vertex of P must be 2. Indeed, by claim 1 this dimension is at least 2, since $f(l^*_i) \neq f(l^*_j)$ for all l^*_i and l^*_j on *L*[5]*
(In particular, this holds for all l^*_i and l^*_j on the same quadruple).
Suppose, by contradiction, that *dim(aff(f(A)))* ≥ 3, for some quadruple *A*, let the quadruple *B, C, D,* and *E* correspond to vertices of P so that A, B, C, D and *E* is a cycle in P.
All the points of *f(B)* and *f(E)* must be at unit distance from those of *f(A)*, so all the points of *f(B)* and *f(E)* lie on a circle, say circle S with enter O.
This means that *f(B)* and *f(C)* are two squares inscribed in S. it follows that all the points of *f(C)* and *f(D)* must lie on the 3-flat that is perpendicular to 2-flat determined by S and passes through O.
But this cannot happen, since the points of *f(C)* span a flat of dimension at least 2 in this 3-flat, which then forces the points of *f(D)* to lie on a line, which is impossible.
It follows that the points of any *f*(F) lie on the intersection of some unit-distance spheres and a 2-flat which is a circle; when F={a, b, c, d} is a given block,
such that $\|a-b\| = \|b-c\| = \|c-d\| = \|d-a\| = 1$ and $\|a-c\| = \|b-d\| = \sqrt{2}$.
Thus *f(a), f(b), f(c),*and *f(d)* form the vertex set of quadrangle, of edge length one that lies in a circle. (See figure 5).

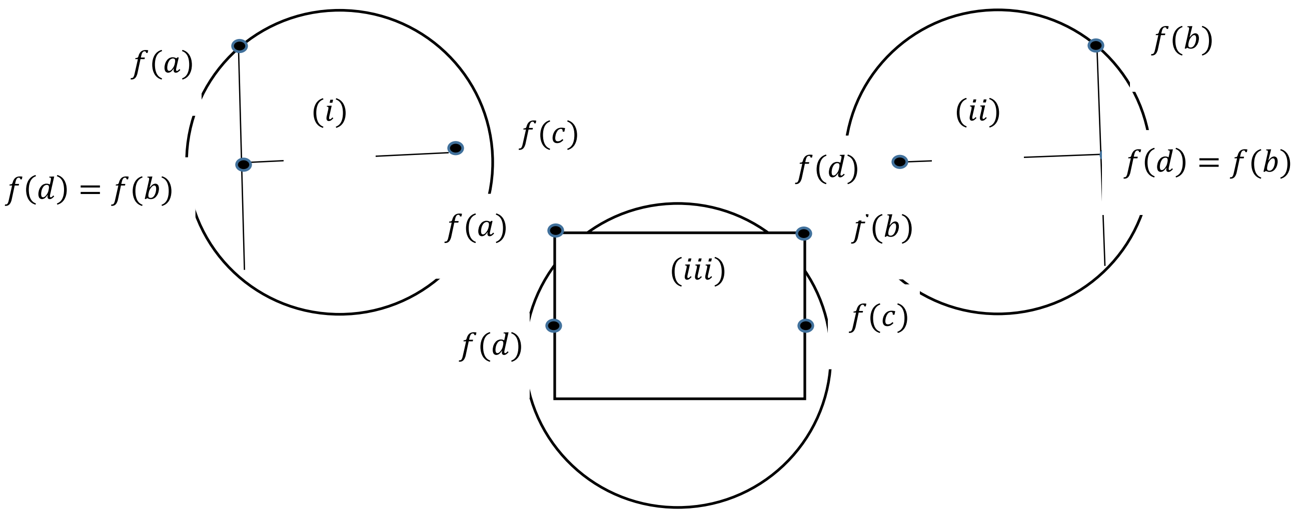

Figure 5

The situations (*i*) and (*ii*) are impossible since $f(l^*_i) \neq f(l^*_j)$ for all $l^*_i$and l^*_j on *L*[5]*.
It follows that *f(a), f(b), f(c),* and *f(d)* form vertex set of a square in circle of diameter $\sqrt{2}$, implying: $\|f(a)-f(c)\| = \|f(b)-f(d)\| = \sqrt{2}$.
Hence, the distance$\sqrt{2}$, within each quadrangle are preserved. In particular
$\|f(Z)-f(W)\| = \sqrt{2}$.
This completes the proof of Theorem 1*.

Proof of Theorem 1:
Let *f* be a unit distance preserving mapping $f{:}Q^5 \to Q^5$. By Theorem 1* the unit distance preserving mapping f preserves the distance $\sqrt{2}$ **.**
Our result follows by using a Theorem of J. Zaks [8], which states that if a mapping
$g{:}Q^d \to Q^d$ preserves the distances 1 and $\sqrt{2}$**,** then *g* is an isometry, provided $d \geq 5$.
<u>Moreover, dim(*aff(f(L[5]))*) = 5:</u>
The mapping f is an isometry, hence it suffices to provide that dim(*aff (L[5])*) = 5.
To show this, notice that:

$$\frac{1}{2}\left(\frac{1}{2},\frac{1}{2},0,0,0\right)+\frac{1}{2}\left(\frac{1}{2},-\frac{1}{2},0,0,0\right)=\frac{1}{2}(1,0,0,0,0)$$

$$\frac{1}{2}\left(\frac{1}{2},\frac{1}{2},0,0,0\right)+\frac{1}{2}\left(-\frac{1}{2},\frac{1}{2},0,0,0\right)=\frac{1}{2}(0,1,0,0,0)$$
$$\frac{1}{2}\left(0,0,\frac{1}{2},\frac{1}{2},0\right)+\frac{1}{2}\left(0,0,\frac{1}{2},-\frac{1}{2},0\right)=\frac{1}{2}(0,0,1,0,0)$$
$$\frac{1}{2}\left(0,0,\frac{1}{2},\frac{1}{2},0\right)+\frac{1}{2}\left(0,0,-\frac{1}{2},\frac{1}{2},0\right)=\frac{1}{2}(0,0,0,1,0)$$
$$\frac{1}{2}\left(0,0,0,\frac{1}{2},\frac{1}{2}\right)+\frac{1}{2}\left(0,0,0,-\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}(0,0,0,0,1)$$

Hence all the major unit vectors in R^5 when multiplied by $\frac{1}{2}$, are convex combinations of points in *L[5]*.

This completes the proof of Theorem 1.

References

1. F.S Beckman and D.A Quarles: On isometries of Euclidean spaces, Proc. Amer. Math. Soc. 4, (1953), 810-815.
2. W.Benz, An elementary proof of the Beckman and Quarles, Elem.Math. 42 (1987), 810-815
3. W.Benz, Geometrische Transformationen, B.I.Hochltaschenbucher, Manheim 1992.
4. Karin B. Chilakamarri: Unit-distance graphs in rational n-spaces Discrete Math. 69 (1988), 213-218.
5. R.Connelly and J.Zaks: The Beckman-Quarles theorem for rational d-spaces, d even and d≥6. Discrete Geometry, Marcel Dekker, Inc. New York (2003) 193-199, edited by Andras Bezdek.
6. H.Lenz: Der Satz von Beckman-Quarles in rationalen Raum, Arch. Math. 49 (1987), 106-113.
7. I.M.Niven, H.S.Zuckerman, H.L.Montgomery: An introduction to the theory of numbers, J. Wiley and Sons, N.Y., (1992).
8. A.Tyszka: A discrete form of the Beckman-Quarles theorem for rational eight- space. Aequationes Math. 62 (2001), 85-93.
9. J.Zaks: A distcrete form of the Beckman-Quarles theorem for rational spaces. J. of Geom. 72 (2001), 199-205.
10. J.Zaks: The Beckman-Quarles theorem for rational spaces. Discrete Math. 265 (2003), 311-320.
11. J.Zaks: On mapping of Q^d to Q^d that preserve distances 1 and $\sqrt{2}$. and the Beckman-Quarles theorem. J of Geom. 82 (2005), 195-203.