The Bienergy of Pull-Back Vector Fields

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Abstract: The problem studied in this paper is related to the bienergy of a pull-back vector field from a Riemannian manifold (M, g) to its tangent bundle TN equipped with the Sasaki metric h^s. We show that a pull-back vector field on a compact manifold (M, g) which covers harmonic map φ. then the pull-back bundle $V \in \Gamma(\varphi^{-1}(TN))$ is biharmonic if and only if V is parallel.
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1. Introduction

Let $\varphi: M \to N$ be a smooth map between the smooth manifolds M, N. The map φ induces the pull-back vector field $V: M \to N$. In the case where M, N are Riemannian manifolds and TN the tangent bundle equipped with the Sasaki metric. The motivation of this paper is to study harmonicity and biharmonicity of the pull-back vector field $V: (M, g) \to (TN, h^s)$.

In this paper, we deal with these problems. We show that if (M, g) is a compact oriented m-dimensional Riemannian manifold and the map φ is harmonic, then the pull-back vector field $V \in \Gamma(\varphi^{-1}(TN))$ is harmonic if and only if V is parallel.

In the biharmonicity, we show that if (M, g) be a compact oriented m-dimensional Riemannian manifold and the map φ is harmonic, then the pull-back vector field $V \in \Gamma(\varphi^{-1}(TN))$ is biharmonic if and only if V is harmonic.

1.2 Harmonic maps

Consider a smooth map $\phi: (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the energy functional is defined by

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 v_g$$

(or over any compact subset $K \subset M$).

A map is called harmonic if it is a critical point of the energy functional E (or $E(K)$ for all compact subset $K \subset M$). For any smooth variation $\{\phi\}_{t\in I}$of ϕ with $\phi_0 = \phi$ and $V = \left.\frac{d\phi_t}{dt}\right|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)|_{t=0} = -\int_M h(\tau(\phi), V)v_g$$

Where

$$\tau(\phi) = tr_g \nabla d\phi$$

Is the tension field of ϕ. Then we have

Theorem 1.1

A smooth map $\phi: (M^m, g) \to (N^n, h)$ is harmonic if and only if

$$\tau(\phi) = 0$$

If $(x^i)_{1\le i\le m}$ and $(y^\alpha)_{1\le \alpha\le n}$ denote local coordinates on M and N respectively then

$$\tau(\phi)^\alpha = \left(\Delta\phi^\alpha + g^{ij}\Gamma_{\beta\gamma}^{\alpha}{}^N \frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\phi^\gamma}{\partial x^j}\right) = 0$$

Where $\boldsymbol{\Delta\phi^{\alpha}} = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}(\sqrt{|g|}g^{ij}\frac{\partial\phi^{\alpha}}{\partial x^j})$ is the Laplace operator on $(\boldsymbol{M^m}, \boldsymbol{g})$ and $\boldsymbol{\Gamma_{\beta\gamma}^{\alpha}}^{\boldsymbol{N}}$ are the Christoffel symbols on $\boldsymbol{N}$.

1.2.Biharmonic maps

Definition1.2.1

A map $\phi: (M^m, g) \to (N^n, h)$ between Riemannian manifolds is called biharmonic if is a critical point of bienergy functional

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g$$

We have

$$\frac{d}{dt}E_2(\phi_t)|_{t=0} = -\int_M h(\tau_2(\phi), V)v_g$$

The Euler-Lagrange equation attached to bienergy is given by the vanishing of the bitension field

$$\tau_2(\phi) = -J_\phi\big(\tau(\phi)\big) = -(\Delta^\phi\tau(\phi) + tr_g R^N(\tau(\phi), d\phi)d\phi)$$

Where J_ϕ is the Jacobi operator defined by

$$J_\phi: \Gamma\big(\phi^{-1}(TN)\big) \to \Gamma\big(\phi^{-1}(TN)\big)$$
$$V \longmapsto \Delta^\phi V + tr_g R^N(V, d\phi)d\phi$$

2.Basic Notions and Definitions on TM

Let (M, g) be an n-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1,\dots,n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1,\dots,n}$ on TM. Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on TM, the vertical distribution V and the horizontal distribution H, defined by

$$V_{(x,u)} = \ker\big(d\pi_{(x,y)}\big)$$
$$= \left\{a^i \frac{\partial}{\partial y^i|_{(x,u)}} \; ; \; a^i \in \mathbb{R}\right\}$$
$$H_{(x,u)} = \left\{\frac{\partial}{\partial x^i|_{(x,u)}} - a^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k|_{(x,u)}}; \; a^i \in \mathbb{R}\right\}$$

Where$(x, u) \in TM$, such that $T_{(x,u)}TM = H(x, u) \oplus V(x; u)$.

Let $X = X^i\frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i\frac{\partial}{\partial y^i}$$

$$X^H = X^i\frac{\delta}{\delta X^i} = X^i\left\{\frac{\partial}{\partial X^i} - y^j\Gamma_{ij}^k\frac{\partial}{\partial y^k}\right\}$$

For consequences, we have $\left(\frac{\partial}{\partial x^i}\right)^H = \frac{\delta}{\delta x^i}$ and$\left(\frac{\partial}{\partial x^i}\right)^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1\dots n}$ is a local adapted frame in TTM

Definition2.1

The Sasaki metric g^s on the tangent bundle TM of M is given by

$$\begin{cases} g^s(X^H, Y^H) = g(X, Y) \circ \pi \\ g^s(X^H, Y^V) = 0 \\ g^s(X^V, Y^V) = g(X, Y) \circ \pi \end{cases}$$

For all vector fields $X, Y \in \Gamma(\mathrm{TM})$.

2.1Proposition

Let (M, g) be a Riemannian manifold and $\widehat{\nabla}$ be the levi-Civita connection of the tangent bundle (TM, g^s) equipped with the Sasaki metric. Then

$$(\widehat{\nabla}_{X^H}Y^H)_{(x,u)} = (\nabla_X Y)^H_{(x,u)} - \frac{1}{2}(R_x(X,Y)u)^V$$
$$(\widehat{\nabla}_{X^H}Y^V)_{(x,u)} = (\nabla_X Y)^V_{(x,u)} + \frac{1}{2}(R_x(u,Y)X)^H$$
$$(\widehat{\nabla}_{X^V}Y^H)_{(x,u)} = \frac{1}{2}(R_x(u,X)Y)^H$$
$$(\widehat{\nabla}_{X^V}Y^V)_{(x,u)} = 0$$

For all vector fields $X, Y \in \Gamma(\mathrm{TM})$ and $(x, u) \in \mathrm{TM}$.

3.1 Lemma

Let (M, g) be a Riemannian manifold. If $X, Y \in \Gamma(TM)$ are vector fields on M and $(x, u) \in TM$ such that $X_x = u$, then we have

$$d_x X(Y_x) = Y^H_{(x,u)} + (\nabla_Y X)^V_{(x,u)}$$

3.2 Lemma

Let $\varphi: (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifold. The map φ induces the pull-back vector fields

$$V: (M, g) \to (TN, h^S)$$
$$x \longmapsto (\varphi(x), Y_{\varphi(x)})$$

For all vector field $V \in \Gamma(\varphi^{-1}(TN))$ and $X \in \Gamma(TM)$, we have

$$dV(X) = (d\varphi(X))^H + (\nabla^{\varphi}_X V)^V$$

Proof

From the Lemma 3.1, we have

$$dV(X_x) = d(Y \circ \varphi)(X_x) = dY\big(d\varphi(X_x)\big)$$
$$= \big(d\varphi(X_x)\big)^H_{(x,u)} + \big(\nabla_{d\varphi(X_x)} Y \circ \varphi\big)^V_{(x,u)}$$
$$= \big(d\varphi(X_x)\big)^H_{(x,u)} + (\nabla^{\varphi}_X V)^V_{(x,u)}$$

3.1. Proposition

The tension field of the pull-back vector fields $V \in \Gamma(\varphi^{-1}(TN))$ is given by

$$\tau(V) = \big(\tau(\varphi) + tr_g R^N\big(V, \nabla^{\varphi}_* V\big)d\varphi(*)\big)^H + \big(tr_g(\nabla^{\varphi 2} V)\big)^V$$

Proof

Let $x \in M$ and $\{e_1\}^n_{i=1}$ be a local orthonormal frame on M such that $\nabla_{e_i} e_j = 0$ at x and $X_x = u$, then by summing over i, we have

$$\tau(V) = \big\{\nabla^V_{e_i} dV(e_i)\big\}$$
$$= \Big\{\nabla^{TN}_{(d\varphi(e_i))^H}\big(d\varphi(e_i)\big)^H + \nabla^{TN}_{(d\varphi(e_i))^H}\big(\nabla^{\varphi}_{e_i} V\big)^V$$
$$+ \nabla^{TN}_{(\nabla^{\varphi}_{e_i} V)^V}\big(\nabla^{\varphi}_{e_i} V\big)^V + \nabla^{TN}_{(\nabla^{\varphi}_{e_i} V)^V}\big(d\varphi(e_i)\big)^H\Big\}$$

$$= \left(\nabla_{d\varphi(e_i)} d\varphi(e_i)\right)^H - \frac{1}{2}\left(R\left(d\varphi(e_i), d\varphi(e_i)\right)V\right)^V$$
$$+\left(\nabla_{d\varphi(e_i)} \nabla^{\varphi}_{e_i} V\right)^V + \frac{1}{2}\left(R\left(V, \nabla^{\varphi}_{e_i} V\right) d\varphi(e_i)\right)^H$$
$$+\tfrac{1}{2}\left(R\left(V, \nabla^{\varphi}_{e_i} V\right) d\varphi(e_i)\right)^H$$

Then

$$\tau(V) = \left(\tau(\varphi) + tr_g R^N\left(V, \nabla^{\varphi}_{*} V\right) d\varphi(*)\right)^H + \left(tr_g(\nabla^{\varphi 2} V)\right)^V$$

3.1. Theorem

The pull-back vector fields $V \in \Gamma(\varphi^{-1}(TN))$ is harmonic if and only if

$$\tau(\varphi) = 0, \quad tr_g R^N\left(V, \nabla^{\varphi}_{*} V\right) d\varphi(*) = 0 \text{ and } tr_g \nabla^{\varphi 2} V = 0$$

3.3Lemma

Let $\varphi: (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. Then the energy density associated to $V \in \Gamma(\varphi^{-1}TN)$ is given by

$$e(V) = e(\varphi) + \frac{1}{2} trace_g h(\nabla^{\varphi}_{*} V, \nabla^{\varphi}_{*} V)$$

Where $e(\varphi)$ is the energy density of the map φ.

Proof

Let $\{e_i, \dots, e_m\}$ be a local orthonormal frame on M, then

$$2e(V) = \sum_{i=1}^{m} h^s(dV(e_i), dV(e_i))$$

Using Lemma 3.2, we obtain

$$2e(V) = \sum_{i=1}^{m} \left\{ h^s(dV(e_i)^H, dV(e_i)^H) + h^s\left(\left(\nabla^{\varphi}_{e_i} V\right)^V, \left(\nabla^{\varphi}_{e_i} V\right)^V\right)\right\}$$
$$= \sum_{i=1}^{m} \left\{ h\ \left(dV(e_i)\ \ , dV(e_i)\ \ \right) + h\ \left(\left(\nabla^{\varphi}_{e_i} V\right)\ \ , \left(\nabla^{\varphi}_{e_i} V\right)\ \ \right)\right\}$$
$$= 2e(\varphi) + h\ \left(\left(\nabla^{\varphi}_{e_i} V\right)\ \ , \left(\nabla^{\varphi}_{e_i} V\right)\ \ \right)$$

3.2Theorem

Let (M, g) be a compact oriented m-dimensional Riemannian manifold and $\varphi: (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. if φ is harmonic, then the pull-back vector fields $V \in \Gamma(\varphi^{-1}(TN))$ is harmonic if and only if V is parallel.

Proof

If φ is harmonic and V is parallel, we deduce that V is harmonic.Inversely:
Let V_t be a compactly supported variation of V defined by $V = (1 + t)V$. From Lemma 3.3, we have

$$e(V_t) = e(\varphi) + \frac{(1+t)^2}{2} trace_g h(\nabla^{\varphi}_{*} V, \nabla^{\varphi}_{*} V)$$

If V is a critical point of the energy functional, then we have:

$$0 = \frac{d}{dt} E(V_t)|_{t=0}$$
$$= \frac{d}{dt}\left(\int_M \left(e(\varphi) + \frac{(1+t)^2}{2} trace_g h\left(\nabla^{\varphi} V, \nabla^{\varphi} V\right)\right) dv_g\right)_{t=0}$$
$$= \frac{d}{dt}\int_M trace_g h\left(\nabla^{\varphi} V, \nabla^{\varphi} V\right) dv_g$$

Then $h\left(\nabla^{\varphi} V, \nabla^{\varphi} V\right) = 0$

4.Biharmonicity of Pull-Back Vector Fields

In this section, we denote

$$\Delta^{\varphi}V = -\mathrm{trace}_g \nabla^{\varphi 2} = \sum_{i=1}^{m}\left\{\nabla^{\varphi}_{\nabla_{e_i}e_i}V - \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}V\right\}$$

$$S(V) = -\sum_{i=1}^{m} R^N\left(V, \nabla^{\varphi}_{e_i}V\right)d\varphi(e_i)$$

Then, we have

$$\tau(V) = (\tau(\varphi) - S(V))^H + (-\Delta^{\varphi}(V))^V$$

4.1.Theorem

Let (M,g) be a compact oriented m-dimensional Riemannian manifold and $V \in \Gamma(\varphi^{-1}(TN))$. Then, we have

$$\frac{d}{dt}E_2(V_t)|_{t=0}\int_M \Bigg\{h\Bigg(\Delta^{\varphi}\Delta^{\varphi}V + \sum_{i=1}^{m}\Big[\left(\nabla^{\varphi}_{e_i}R\right)\left(e_i, S(V)\right)V$$
$$+R\left(e_i, \nabla^{\varphi}_{e_i}S(V)\right)V + 2R\left(e_i, S(V)\right)\nabla^{\varphi}_{e_i}V$$
$$-\left(\nabla^{\varphi}_{e_i}R\right)\left(e_i, \tau(\varphi)\right)V - R\left(e_i, \nabla^{\varphi}_{e_i}\tau(\varphi)\right)V$$
$$-2R\left(e_i, \tau(\varphi)\right)\nabla^{\varphi}_{e_i}V, V\Big) + h\left(R\left(S(V), d\varphi(e_i)\right)d\varphi(e_i)\right)$$
$$+\Delta^{\varphi}S(V) - \tau_2(\varphi), v-\Bigg)\Bigg]\Bigg\}v_g$$

For any smooth 1-parameter variation $U: M \times (-\epsilon, \epsilon) \xrightarrow{\phi} N \xrightarrow{Y} TN$ of V through vector fields i.e. $V_t(z) = Y \circ \phi(z,t) = U(t,z) \in T_{\varphi(z)}N$ for any $|t| < \epsilon$ and $z \in M$, or equivalently $V_t \in \Gamma(\varphi^{-1}(TN))$ for any $|t| < \epsilon$.

Also, W is the tangent vector field on M given by

$$W(z) = \frac{d}{dt}V_z(0), \quad z \in M.$$

Where $V_z(t) = U(z,t), \quad (z,t) \in M \times (-\epsilon, \epsilon)$.

Proof

Let $V \in \Gamma(\varphi^{-1}(TN))$ and $I = (-\epsilon, \epsilon), \epsilon > 0$. For $t \in I$, we denote by $i_t: M \to M \times I, p \to (p,t)$ the canonical injection. We consider C^{∞}-variations $U: M \times I \to TN$ of V, i.e. for all $t \in I$ the mapping $V_t = U \circ i_t$ are in fact vector field and $V_0 = V$. We choose $\{e_i\}_{i=1}^m$ a local orthonormal frame field of (M,g).

We extend e_i (resp.$\frac{d}{dt} \in \Gamma(I)$) to $M \times I$, denoted by E_i (resp.$\frac{d}{dt}$). Moreover, we have $\left[E_i, \frac{d}{dt}\right] = 0$. We denote by D^{Φ} the pull-back Levi-Civita connection of $M \times I$ and R^D the pull-back Riemann curvature tensor of $M \times I$. Since $M \times I$ is a Riemannian product, we have (using the second Bianchi identity for the last relation)

$$R^D(TN, TI) = 0, \quad D^{\phi}_{\frac{d}{dt}}d\phi(E_i) = 0, \quad D^{\phi}_{E_i}d\phi\left(\frac{d}{dt}\right) = 0, \quad (D^{\phi}_{\frac{d}{dt}}R^D)\left(D^{\phi}_{E_i}U, U\right)d\phi(E_i) = 0$$

For all $1 \leq i \leq$ m. We set $Z = \sum_{i=1}^m R^D\left(D^{\phi}_{E_i}U, U\right)d\phi(E_i)$ and $\Omega = \sum_{i=1}^m\left[D^{\phi}_{D_{E_i}E_i}U - D^{\phi}_{E_i}D^{\phi}_{E_i}\right]$. We easily observe that $S(V_t) = Z \circ i_t$ and $\Delta^{\varphi}V_t = \Omega \circ i_t$. In the sequel, we consider the function

$$E_2(V_t) = \frac{1}{2}\int_M \left[h\left(\tau(\varphi), \tau(\varphi)\right) + h(S(V_t), S(V_t)) - 2h(S(V_t), \tau(\varphi))\right.$$
$$\left.+h(\Delta^{\varphi}V_t, \Delta^{\varphi}V_t)\right]v_g$$
$$= \frac{1}{2}\int_M \left[h\left(\tau(\varphi), \tau(\varphi)\right) + h(Z,Z) - 2h\left(Z, \tau(\varphi)\right) + h(\Omega, \Omega)\right] \circ i_t v_g$$

Differentiating the function $E_2(V_t)$ at each t, we obtain

$$\frac{d}{dt}E_2(V_t) = \int_M h\left(D^{\phi}_{\frac{d}{dt}}\tau(\varphi), \tau(\varphi)\right) \circ i_t v_g + \int_M h\left(D^{\phi}_{\frac{d}{dt}}Z, Z\right) \circ i_t v_g$$
$$-\int_M h\left(D^{\phi}_{\frac{d}{dt}}Z, \tau(\varphi)\right) \circ i_t v_g + \int_M h\left(D^{\phi}_{\frac{d}{dt}}\Omega, \Omega\right) \circ i_t v_g$$
$$-\int_M h(Z, D^{\phi}_{\frac{d}{dt}}\tau(\varphi)) \circ i_t v_g$$

Taking into account the symmetries of the Riemann curvature tensor and summing over all repeated indices, we have

$$\int_M h\left(D^{\phi}_{\frac{d}{dt}} Z, Z\right) \circ i_t v_g = \int_M \left(\left(D^{\phi}_{\frac{d}{dt}} R^D\right)\left(D^{\phi}_{\frac{d}{dt}} U, U\right) d\phi(E_i) + R^D\left(D^{\phi}_{\frac{d}{dt}} D^{\phi}_{\frac{d}{dt}} U, U\right) d\phi\,(E_i)\right.$$
$$\left. + R^D\left(D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{\frac{d}{dt}} U\right) d\phi(E_i), Z\right) \circ i_t v_g$$
$$= \int_M \left[h\left(R^D\left(D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} U + R^{D^{\phi}}\left(\frac{d}{dt}, E_i\right) U, U\right) d\phi(E_i), Z\right)\right.$$
$$\left. + h(R^D(d\phi(E_i), Z) D^{\phi}_{E_i} U, D^{\phi}_{\frac{d}{dt}} U)\right] \circ i_t v_g$$
$$= \int_M \left[-h(R^D(d\phi(E_i), Z) U, D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} U)\right.$$
$$\left. + h(R^D(d\phi(E_i), Z) D^{\phi}_{E_i} U, D^{\phi}_{\frac{d}{dt}} U\right] \circ i_t v_g$$
$$= \int_M \left\{-D^{\phi}_{E_i}\left(h(R^D(d\phi(E_i), Z) U, D^{\phi}_{\frac{d}{dt}} U)\right)\right.$$
$$+ h(R^D\left(D^{\phi}_{E_i} d\phi(E_i), Z\right) U, D^{\phi}_{\frac{d}{dt}} U)$$
$$+ h\left((D^{\phi}_{E_i} R^D)(d\phi(E_i), Z) U, D^{\phi}_{\frac{d}{dt}} U\right)$$
$$+ h(R^D\left(d\phi(E_i), D^{\phi}_{E_i} Z\right) U, D^{\phi}_{\frac{d}{dt}} U)$$
$$\left. + 2h(R^D(d\phi(E_i), Z) D^{\phi}_{E_i} U, D^{\phi}_{\frac{d}{dt}} U)\right\} \circ i_t v_g$$

Applying the divergence Theorem for the 1-form

$$\eta_t(W) = h\left(R\big(d\varphi(W), S(V_t)\big) V_t, \nabla^{\varphi}_{\frac{d}{dt}} V_t\right), \qquad t \in I, \quad d\varphi(W) \in \Gamma(\varphi^{-1}(TN))$$

Then

$$\int_M h(D^{\phi}_{\frac{d}{dt}} Z, Z) \circ i_t v_g = \int_M h\left((\nabla^{\varphi}_{e_i} R)\big(d\varphi(e_i), S(V_t)\big) V_t + R(d\varphi(e_i), \nabla^{\varphi}_{e_i} S(V_t)) V_t\right.$$
$$\left. + 2R\big(d\varphi(e_i), S(V_t)\big) \nabla^{\varphi}_{e_i} V_t, \nabla^{\varphi}_{\frac{d}{dt}} V_t\right) v_g$$

Similarly, summing over all repeated indices, we deduce

$$\int_M h(\, D^{\phi}_{\frac{d}{dt}} \Omega, \Omega) \circ i_t v_g = \int_M h(\, D^{\phi}_{\frac{d}{dt}} D^{\phi}_{D_{E_i} E_i} U - D^{\phi}_{\frac{d}{dt}} D^{\phi}_{E_i} D^{\phi}_{E_i} U, \Omega) \circ i_t v_g$$
$$= \int_M h(\, D^{\phi}_{D_{E_i} E_i} D^{\phi}_{\frac{d}{dt}} U - D^{\phi}_{E_i} D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} U, \Omega) \circ i_t v_g$$
$$= \int_M \left\{D^{\phi}_{D_{E_i} E_i}[h(D^{\phi}_{\frac{d}{dt}}, \Omega)] - h(D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{D_{E_i} E_i} \Omega)\right.$$
$$\left. - D^{\phi}_{E_i}[h(D^{\phi}_{\frac{d}{dt}} U, \Omega) + h(D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{E_i} \Omega)\right\} \circ i_t v_g$$
$$= \int_M \left\{D^{\phi}_{D_{E_i} E_i}[h(\, D^{\phi}_{\frac{d}{dt}} U, \Omega)] - D^{\phi}_{E_i} D^{\phi}_{E_i}[h(D^{\phi}_{\frac{d}{dt}} U, \Omega)]\right.$$
$$- h\left(D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{D_{E_i} E_i} \Omega\right) + D^{\phi}_{E_i}[h(D^{\phi}_{\frac{d}{et}} U, D^{\phi}_{E_i} \Omega)$$
$$\left. + h\left(D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{E_i} \Omega\right)\right\} \circ i_t v_g$$

$$= \Delta^{\phi}[h(D^{\phi}_{\frac{d}{dt}}U,\Omega)] - h(D^{\phi}_{\frac{d}{dt}}U, D^{\phi}_{D_{E_i}E_i}\Omega)$$
$$\left.+2D^{\phi}_{E_i}[h(D^{\phi}_{\frac{d}{dt}}U, D^{\phi}_{E_i}\Omega) - h(D^{\phi}_{\frac{d}{dt}}U, D^{\phi}_{E_i}D^{\phi}_{E_i}\Omega)\right\} \circ i_t v_g$$

$$= \Delta^{\phi}[h(D^{\phi}_{\frac{d}{dt}}U,\Omega)] - h(D^{\phi}_{\frac{d}{dt}}U, D^{\phi}_{D_{E_i}E_i}\Omega)$$
$$+2D^{\phi}_{E_i}[h(D^{\phi}_{\frac{d}{dt}}U, D^{\phi}_{E_i}\Omega) - 2h(D^{\phi}_{\frac{d}{dt}}U, D^{\phi}_{D_{E_i}E_i}\Omega)$$
$$\left.+2h\left(D^{\phi}_{\frac{d}{dt}}U, D^{\phi}_{D_{E_i}E_i}\Omega\right) - h(D^{\phi}_{\frac{d}{dt}}U, D^{\phi}_{E_i}D^{\phi}_{E_i}\Omega)\right\} \circ i_t v_g$$

Applying the divergence Theorem for the 1-form

$$\theta_t(.) = h\left(\nabla^{\varphi}_{\frac{d}{dt}}V_t, \nabla^{\varphi}\Delta^{\varphi}V_t\right), \qquad t \in I$$

We have

$$\int_M h(\,D^{\phi}_{\frac{d}{dt}}\Omega,\Omega) \circ i_t v_g = \int_M \Delta^{\phi} h(D^{\varphi}_{\frac{d}{dt}}V_t, \Delta V_t)]v_g + 2\int_M div(\theta_t)v_g$$
$$+ \int_M h\left(\nabla^{\varphi}_{\frac{d}{dt}}V_t, \nabla^{\varphi}\Delta^{\varphi}V_t\right)v_g$$
$$= \int_M h\left(\nabla^{\varphi}_{\frac{d}{dt}}V_t, \nabla^{\varphi}\Delta^{\varphi}V_t\right)v_g$$

Similarly, summing over all repeated indices, we deduce

$$\int_M h(D^{\phi}_{\frac{d}{dt}}Z, \tau(\varphi)) \circ i_t v_g = \int_M h\left(\left(D^{\phi}_{\frac{d}{dt}}R^D\right)\left(D^{\phi}_{E_i}U, U\right)d\phi(E_i) + R^D\left(D^{\phi}_{\frac{d}{dt}}D^{\phi}_{E_i}U, U\right)d\phi(E_i)\right.$$
$$\left.+ R^D\left(D^{\phi}_{E_i}U, D^{\phi}_{\frac{d}{dt}}U\right)d\phi(E_i), \tau(\phi)\right) \circ i_t v_g$$
$$= \int_M \left[h(R^D\left(D^{\phi}_{E_i}D^{\phi}_{\frac{d}{dt}}U + R^D \quad \left(\frac{d}{dt}, E_i\right)U, U)d\phi(E_i), \tau(\phi)\right)\right.$$
$$\left. + h(R^D\big(d\phi(E_i), \tau(\phi)\big)D^{\phi}_{E_i}U, D^{\phi}_{\frac{d}{dt}}U)\right] \circ i_t v_g$$
$$= \int_M \left[-h(R^D\big(d\phi(E_i), \tau(\phi)\big)U, D^{\phi}_{E_i}D^{\phi}_{\frac{d}{dt}}U)\right.$$
$$\left. + h(R^D\big(d\phi(E_i), \tau(\phi)\big)D^{\phi}_{E_i}U, D^{\phi}_{\frac{d}{dt}}U)\right] \circ i_t v_g$$
$$= \int_M \left\{-D^{\phi}_{E_i}\left(h\left(R^D\big(d\phi(E_i), \tau(\phi)\big)U, D^{\phi}_{\frac{d}{dt}}U\right)\right)\right.$$
$$+h(R^D\left(D^{\phi}_{E_i}d\phi(E_i), \tau(\phi)\right)U, D^{\phi}_{\frac{d}{dt}}U)$$
$$+h\left(\left(D^{\phi}_{E_i}R^D\right)\big(d\phi(E_i), \tau(\phi)\big)U, D^{\phi}_{\frac{d}{dt}}U\right)$$
$$+h(R^D\left(d\phi(E_i), D^{\phi}_{E_i}\tau(\phi)\right)U, D^{\phi}_{\frac{d}{dt}}U)$$
$$\left.+2h(R^D\big(d\phi(E_i), \tau(\phi)\big)D^{\phi}_{E_i}U, D^{\phi}_{\frac{d}{dt}}U)\right\} \circ i_t v_g$$

Applying the divergence Theorem for the 1-form

$$\eta_t(W) = h\left(R(d\varphi(W), \tau(\phi))V_t, \nabla^{\varphi}_{\frac{d}{dt}}V_t\right), \qquad t \in I, \quad d\varphi(W) \in \Gamma(\varphi^{-1}(TN))$$

Then

$$\int_M h(D^{\phi}_{\frac{d}{dt}} Z, \tau(\phi)) \circ i_t v_g = \int_M h\Big((\nabla^{\varphi}_{e_i} R)(d\varphi(e_i), \tau(\phi))V_t + R(d\varphi(e_i), \nabla^{\varphi}_{e_i}\tau(\phi))V_t$$
$$+2R(d\varphi(e_i), \tau(\phi))\nabla^{\varphi}_{e_i} V_t, \nabla^{\varphi}_{\frac{d}{dt}} V_t\Big) v_g$$

From Definition 1.1, we have

$$\int_M h(D^{\phi}_{\frac{d}{dt}} \tau(\phi), \tau(\phi))|_{t=0} \circ i_t v_g = -\int_M h(\tau_2(\phi), v) v_g$$

Where$v = d\phi(\frac{d}{dt})$.

Similarly, summing over all repeated indices, we deduce

$$\int_M h(D^{\phi}_{\frac{d}{dt}} \tau(\phi), Z) \circ i_t v_g = \int_M h(D^{\phi}_{\frac{d}{dt}} D^{\phi}_{E_i} d\phi(E_i) - D^{\phi}_{\frac{d}{dt}} d\phi\big(D_{E_i}E_i\big), Z) \circ i_t v_g$$
$$= \int_M h(R\left(d\phi\left(\tfrac{d}{dt}\right), d\phi(E_i)\right) d\phi(E_i) + D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} d\phi(E_i)$$
$$+D^{\phi}_{[\frac{d}{dt}, E_i]} d\phi(E_i) - D^{\phi}_{D_{E_i}E_i} d\phi\left(\tfrac{d}{dt}\right), Z) \circ i_t v_g$$
$$= \int_M \left(h\left(R(d\phi\left(\frac{d}{dt}\right), d\phi(E_i)) d\phi(E_i), Z\right)\right.$$
$$\left.+h(D^{\phi}_{E_i} D^{\phi}_{E_i} d\phi\left(\tfrac{d}{dt}\right), Z)\right) \circ i_t v_g$$
$$= \int_M \left(h(R\big(Z, d\phi(E_i)\big) d\phi(E_i), d\phi\left(\tfrac{d}{dt}\right))\right.$$
$$+E_i(h\left(D^{\phi}_{E_i} d\phi\left(\tfrac{d}{dt}\right), Z\right))$$
$$\left.-E_i\left(h\left(d\phi\left(\frac{d}{dt}\right), D^{\phi}_{E_i} Z\right)\right) + h(D^{\phi}_{E_i} D^{\phi}_{E_i} Z, d\phi\left(\frac{d}{dt}\right))\right) \circ i_t v_g$$

Applying the divergence Theorem for the 1-form

$$\omega(.) = \left(h\left(D^{\phi}_{.} d\phi\left(\frac{d}{dt}\right), Z\right)\right), \qquad \eta(.) = h(d\phi\left(\frac{d}{dt}\right), D^{\phi}_{.} Z)$$

We have

$$\int_M h(D^{\phi}_{\frac{d}{dt}} \tau(\phi), S(V))|_{t=0} \circ i_t v_g = \int_M h(R\big(S(V), d\phi(e_i)\big) d\phi(e_i) - \Delta^{\varphi} S(V), v) v_g$$

Evaluating at $t = 0$ and setting $V = \nabla_{\frac{d}{dt}} V_t|t = 0$, we easily obtain the result of theorem 4.1

Since the bull-back vector field V is biharmonic if and only if $\frac{d}{dt} E_2(V_t)|_{t=0} = 0$.

For all admissible variations, we get

4.1Corollary.

Pull-back vector field V of an m-dimensional Riemannian manifold (M, g) is biharmonic if and only if

$$\Delta^{\varphi}\Delta^{\varphi} V + \sum_{i=1}^{m} \Big[(\nabla^{\varphi}_{e_i} R)\big(e_i, S(V)\big)V + R\left(e_i, \nabla^{\varphi}_{e_i} S(V)\right)V + 2R(e_i, S(V))\nabla^{\varphi}_{e_i} V$$
$$-\big(\nabla^{\varphi}_{e_i} R\big)\left(e_i, \nabla^{\varphi}_{e_i}\tau(\varphi)\right)V - R\left(e_i, \nabla^{\varphi}_{e_i}\tau(\varphi)\right)V - 2R(e_i, \tau(\varphi))\nabla^{\varphi}_{e_i} V$$
$$+R\big(S(V), d\varphi(e_i)\big)d\varphi(e_i) + \Delta^{\varphi} S(V) - \tau_2(\varphi)\Big] = 0$$

4.1. Remark

If a pull-back vector field of a Riemannian manifold (Mg) defines a harmonic map from (M, g) into (TN, h^s) i.e. $S(V) = 0, \tau(\varphi) = 0$ and $\Delta^{\varphi} V = 0$, then it is automatically a biharmonic pull-back vector field.

4.2. Theorem

Let (M, g) be a compact oriented n-dimensional Riemannian manifold and $\varphi: (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. if φ is harmonic. Then the pull-back bundle $V \in \Gamma(\varphi^{-1}(TN))$ is biharmonic if and only if V is parallel.

Proof.

Let V_t be a compactly supported variation of V defined by $V_t = (1+t)V$. Then

$$\Delta^{\varphi} V_t = (1+t)\Delta^{\varphi} V,$$
$$S(V_t) = S(V).$$

$$E_2(V_t) = \frac{1}{2}\int_M h^s\big(\tau(V_t), \tau(V_t)\big)v_g$$
$$= \frac{1}{2}\int_M h(\Delta^{\varphi} V_t, \Delta^{\varphi} V_t)v_g + \frac{1}{2}\int_M h\big(S(V_t), S(V_t)\big)v_g$$
$$= \frac{(1+t)^2}{2}\int_M h(\Delta^{\varphi} V, \Delta^{\varphi} V)v_g + \frac{(1+t)^4}{2}\int_M h\big(S(V), S(V)\big)v_g$$

Since the pull-back vector field V is biharmonic, then for the variation V_t, we have

$$\frac{d}{dt}E_2(V_t)|_{t=0} = \int_M h(\Delta^{\varphi} V, \Delta^{\varphi} V)v_g + 2\int_M h\big(S(V), S(V)\big)v_g = 0$$

Hence

$$\Delta^{\varphi} V = 0 \qquad \text{and} \qquad S(V_t) = 0$$

Then V is harmonic and from Theorem 3.2, the pull-back vector field V is parallel.

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