A Generalized Sign Pattern Matrix

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Abstract: This article is a number of fundamental results are given and various constructions are presented. The sign patterns of symmetric idempotent matrices through order 5 are determined. Sign patterns of idempotent matrices, especially symmetric idempotent matrices are investigated.

Keywords: sign pattern matrix, generalized sign pattern matrix, idempotent matrix, symmetric idempotent matrix, symmetric orthogonal matrix.

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1. Introduction

A matrix whose entries come from the set {+, −, 0} is called a *sign pattern matrix* (or sign pattern, or pattern). We denote the set of all $n \times n$ sign pattern matrices by Q_n. For a real matrix B, by sgn B we mean the sign pattern matrix in which each positive (respectively, negative, zero) entry of B is replaced by + (respectively,−,0). If $A \in Q_n$, then the *sign pattern class of A* is defined by

$$Q(A) = \{B \in M_n(R) \mid \text{sgn } B = A\}.$$

Suppose P is a property referring to a real matrix. Then A is said to require P if every real matrix in Q(A) has property P, or to allow P if some real matrix in Q(A) has property P.

A permutation sign pattern matrix P is obtained by replacing the 1's in a real permutation matrix by + signs. Then $P^T AP$ gives a "permutation similarity" of the pattern A. A signature pattern S is an $n \times n$ diagonal pattern with nonzero diagonal entries. Hence, the product S^2 is an $n \times n$ diagonal pattern with + diagonal entries (indicated subsequently by I_n or I). Then SAS gives a "signature similarity" of the pattern A.

If A is an n x n sign pattern, then A is sign nonsingular if every B $\in$ Q(A) is nonsingular. Sign nonsingular matrices have been heavily studied and it is well known that they have unambiguously signed determinants; that is, there is at least one nonzero term in the determinant, and all nonzero terms have the same sign.

By the minimum rank of an n x n sign pattern matrix A we mean min $_{B\in Q(A)}$ {rank B}, and denote thisby *mr* A.

If A is an $n \times n$ matrix, then A is permutation similar to a Frobenius normal form matrix

$$\begin{pmatrix} A_{11} & & * \\ & \ddots & \\ 0 & & A_{mm} \end{pmatrix}$$

where the (square) diagonal blocks A_{ii} are the irreducible components of A. The one-by-one irreduciblecomponents can be zero blocks.

We introduce the symbol # to represent a qualitatively ambiguous sum, that is, # = (+) + (-). Wenext recall[3] that a generalized sign pattern matrix $\hat{A} = (\hat{a}_{ij})$ is a matrix whose entries are in the set $\{+, -, 0, \#\}$, and $Q(\hat{A}) = \{B = (b_{ij}) \in M_n(R) | b_{ij}$ is arbitrary if $\hat{a}_{ij} = \#$; sgn $b_{ij} = \hat{a}_{ij}$ if $\hat{a}_{ij} \in \{+, -, 0\}\}$. Two generalized sign pattern matrices $\widehat{A_1}$ and $\widehat{A_2}$ are said to be compatible, denoted by $\widehat{A_1} \overset{c}{\leftrightarrow} \widehat{A_2}$, if there exists a matrix $B \in Q(\widehat{A_1}) \cap Q(\widehat{A_2})$.

Hence

$$\widehat{A_1} = \begin{pmatrix} \# & + \\ - & \# \end{pmatrix} \overset{c}{\leftrightarrow} \begin{pmatrix} + & + \\ \# & 0 \end{pmatrix} = \widehat{A_2}$$

As in [2], ID is the class of all square patterns A for which there exists $B \in Q(A)$ where $B^2 = B$ (A allows a real idempotent). We further let SID denote the class of all symmetric patterns A for which there exists symmetric $B \in Q(A)$ where $B^2 = B$ (A allows a real symmetric idempotent). Clearly, $SID \subseteq ID$. Further, if $A \in$ ID, then by necessity, $A^2 \overset{c}{\leftrightarrow} A$ must hold.

The non-negative patterns in ID were characterized[2]. A square sign pattern matrix A is said to be sign idempotent when $A^2 = A$, these patterns were originally discussed [1]. The sign idempotent patterns in ID were recently characterized [4].

2. A Sign Pattern Matrix

Lemma 2.1

Each of the classes ID and SID is closed under the following operations:

(i) permutation similarity,

(ii) signature similarity,

(iii) transposition,

(iv) direct sum,

(v) Kronecker product.

Theorem 2.2

Suppose that $A \in Q_n$ and mrA =1. Then $A \in ID$ if and only if $A^2 \overset{c}{\leftrightarrow} A$.

Proof:

We have already observed that $A \in ID$ implies $A^2 \overset{c}{\leftrightarrow} A$. Now let $A = uv^T$. Then $A^2 \overset{c}{\leftrightarrow} A \Leftrightarrow uv^Tuv^T \overset{c}{\leftrightarrow} uv^T \Leftrightarrow v^Tu \overset{c}{\leftrightarrow} +$. With B=$xy^T$, where sgn x =u, sgn y=v, and $y^Tx = 1$, we have $B^2 = B \in Q(A)$.

Preposition 2.3

Suppose that A is an $n \times n$ irreducible symmetric sign pattern matrix, and mrA = 1.

Then A∈ SID if and only if A is signature similar to J_n.

Proof:

Assume $A \epsilon SID$. Then $A^2 \overset{c}{\leftrightarrow} A$.Since A is irreducible and symmetric, it is then easily seen that all the diagonal entries of A are +. With $mrA = 1$, we have $A = uu^T$, where each entry in u is nonzero.Hence A is signature similar to J_n.

Conversely, suppose A is signature similar to J_n. If B is the n × n matrix each of whose entries is 1/n, then B is a symmetric idempotent in Q(J_n). Thus, $A \epsilon SID$.

Since the only nonsingular idempotent matrix is the identity matrix. It is clear that a symmetric pattern A is in SID if and only if each irreducible component of A is in SID. We let J_n denote the all + pattern of order n.

Proposition 2.4

The only sign nonsingular sign pattern matrix in ID is I_n.

The only 2×2 sign patterns in ID are 0, I_2, or the 2×2 sign patterns A where $mrA = 1$ and $A^2 \overset{c}{\leftrightarrow} A$.

Under equivalence (permutation similarity, signature similarity, and transposition), we find six representatives of the 2×2 sign patterns in ID:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix}, \begin{pmatrix} + & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} + & + \\ + & + \end{pmatrix}, \begin{pmatrix} 0 & + \\ 0 & + \end{pmatrix}, \begin{pmatrix} + & + \\ - & - \end{pmatrix}$$

Clearly, the first four patterns are in SID.

Every $n \times n$ matrix B is a principal submatrix of a $2n \times 2n$ idempotent matrix of rank n, as it is easily checked that

$$\begin{pmatrix} B & B \\ I - B & I - B \end{pmatrix}$$

is idempotent and has trace n.

Lemma 2.5

If $\begin{pmatrix} B & C \\ D & E \end{pmatrix}$ is idempotent, where B is square, then C has at least $min\{ rank(X^2 - X) | X \in Q(sgn\ B)\}$ Columns.

Proof:

The block matrix is idempotent implies $B^2 + CD = B$, so that rank $(B^2 - B) = rank(CD) \leq rankC \leq$ the number of columns of C.

Example 2.6

Consider the $n \times n$ sign pattern matrix

$$A_1 = \begin{pmatrix} 0 & + & & & \\ & 0 & + & & \\ & & 0 & \ddots & \\ & & & \ddots & + \\ & & & & 0 \end{pmatrix}$$

Then, if $\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in ID$, A_2 must have at least n columns. To observe that for each $B \in Q(A_1)$, the eigenvalues of B are the n-th roots of a negative number, so that B has no nonnegative eigenvalue. Hence, B(I-B) is invertible, that is, rank $(B^2 - B)$ = n. That A_2 must have at least n columns.

Lemma 2.7

Let B_1 and X be square matrices, and let k be any positive integer. If the real matrix

$B = \begin{pmatrix} B_1 & U \\ V & X \end{pmatrix}$ is idempotent, then the matrix

$$\hat{B} = \begin{pmatrix} B_1 & \frac{1}{\sqrt{k}}U & \frac{1}{\sqrt{k}}U & \cdots & \frac{1}{\sqrt{k}}U \\ \frac{1}{\sqrt{k}}V & \frac{1}{k}X & \frac{1}{k}X & \cdots & \frac{1}{k}X \\ \frac{1}{\sqrt{k}}V & \frac{1}{k}X & \frac{1}{k}X & \cdots & \frac{1}{k}X \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{k}}V & \frac{1}{k}X & \frac{1}{k}X & \cdots & \frac{1}{k}X \end{pmatrix}$$

of block size $(k+1) \times (k+1)$ is idempotent.

Theorem 2.8

If $\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in ID$, where A_1 is square, then the square pattern

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_2 \\ A_3 & A_4 & \cdots & A_4 \\ \vdots & \vdots & \ddots & \vdots \\ A_3 & A_4 & \cdots & A_4 \end{pmatrix} \in ID$$

Theorem 2.9

If $\begin{pmatrix} A_1 & A_2 \\ A_2^T & A_4 \end{pmatrix} \in SID$, Where A_1 is square, then the square pattern

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_2 \\ A_2^T & A_4 & \cdots & A_4 \\ \vdots & \vdots & \ddots & \vdots \\ A_3 & A_4 & \cdots & A_4 \end{pmatrix} \in SID$$

3. A Symmetric Pattern Matrix

Lemma 3.1

Let $B = (b_{ij})$ be a real symmetric idempotent matrix. If $b_{ij} \neq 0$ for some i and j, i≠j, then 0< $b_{ii}, b_{jj} < 1$ and $|b_{ij}| \leq \min\{\frac{1}{2}, \sqrt{b_{ii}b_{jj}}\}$.

Proof:

Comparing the (i,i) entries of B = B^2 , we see that

$$b_{ii} - b_{ii}^2 = b_{ij}^2 + \sum_{\substack{k\neq i \\ k\neq j}} b_{ik}^2 > 0$$

If follows that 0< $b_{ii} < 1$. Similarly , we are 0< $b_{jj} < 1$. The above equation also shows that $b_{ij}^2 \leq b_{ii} - b_{ii}^2$. since the maximum of the quadratic function $x - x^2$ is $\frac{1}{4}$, we see that $|b_{ij}| \leq \frac{1}{2}$.

Finally, the matrix $\begin{pmatrix} b_{ii} & b_{ij} \\ b_{ij} & b_{jj} \end{pmatrix}$ is positive semidefinite, since it is a principal submatrix of the positive semidefinite matrix B. Hence, det $\begin{pmatrix} b_{ii} & b_{ij} \\ b_{ij} & b_{jj} \end{pmatrix} = b_{ii}b_{jj} - b_{ij}^2 \geq 0$. Therefore, $|b_{ij}| \leq \sqrt{b_{ii}b_{jj}}$.

Theorem 3.2

If $A = \begin{pmatrix} A_1 & A_2 \\ A_2^T & + \end{pmatrix} \in SID$, where A_2 is a nonzero column, then

$$\hat{A} = \begin{pmatrix} A_1 & A_2 & A_2 \\ A_2^T & + & - \\ A_2^T & - & + \end{pmatrix} \in SID$$

Proof:

If $B = \begin{pmatrix} B_1 & Y \\ Y^T & \alpha \end{pmatrix} \in Q(A)$ is a real symmetric idempotent matrix, where 0< $\alpha < 1$ (by lemma 2.1), then it is easily verified that

$$\hat{B} = \begin{pmatrix} B_1 & \frac{1}{\sqrt{2}}Y & \frac{1}{\sqrt{2}}Y \\ \frac{1}{\sqrt{2}}Y^2 & \frac{1}{2}(1+\alpha) & \frac{1}{2}(\alpha-1) \\ \frac{1}{\sqrt{2}}Y^2 & \frac{1}{2}(\alpha-1) & \frac{1}{2}(1+\alpha) \end{pmatrix} \in Q(\hat{A})$$

is symmetric idempotent.

Example 3.3

If $A = \begin{pmatrix} A_1 & A_2 \\ A_2^T & + \end{pmatrix} \in SID$, where A_2 is a nonzero column, then the square pattern

$$\begin{pmatrix} A_1 & A_2 & A_2 & \cdots & A_2 \\ A_2^T & + & - & \cdots & - \\ A_2^T & - & + & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & - \\ A_2^T & - & \cdots & - & + \end{pmatrix} \in SID$$

Example 3.4

For each $1 < k < n$, where $n \geq 3$, the $n \times n$ pattern

$$\begin{pmatrix} J_{n-k} & & J_{n-k,k} & \cdots & \\ & + & - & \cdots & - \\ J_{n-k} & - & + & \ddots & \vdots \\ & \vdots & \ddots & \ddots & - \\ & - & \cdots & - & + \end{pmatrix} \in SID, \qquad \begin{pmatrix} & & & + \\ & J_{n-k} & & \vdots \\ & & & + \\ + & \cdots & + & + \end{pmatrix} \in SID$$

Proposition 3.5

Let

$$A = \begin{pmatrix} + & 0 & 0 & + & + & + & + \\ 0 & + & + & 0 & + & + & - \\ 0 & + & + & 0 & + & - & + \\ + & 0 & 0 & + & + & - & - \\ + & + & + & + & + & + & + \\ + & + & - & - & + & + & + \\ + & - & + & - & + & + & + \end{pmatrix}$$

Then A is irreducible and symmetric, $A^2 \overset{c}{\leftrightarrow} A$, but $A \notin ID$.

Proof

It is easy to check that A is irreducible and symmetric, and $A^2 \overset{c}{\leftrightarrow} A$. Assume that A allows an idempotent matrix

$$B = \begin{pmatrix} & & & & & a & b \\ & & & & & * & * \\ & & B_1 & & & * & * \\ & & & & & -g & -h \\ & & & & & * & * \\ * & c & -e & * & * & * & * \\ * & -d & f & * & * & * & * \end{pmatrix}$$

Where B_1 is 5×5 and the * entries are immaterial. Equating the (1,2),(1,3),(4,2) and (4,3) entries in $B^2 = B$, we get

$$s_1 + ac - bd = 0$$

$$s_2 - ae + bf = 0$$

$$s_3 - cg + dh = 0$$

$$s_4 + eg - fh = 0$$

For some $s_i > 0,\ 1 \le i \le 4$. Hence,

$$bd > ac \qquad (1)$$

$$ae > bf \qquad (2)$$

$$cg > dh \qquad (3)$$

$$fh > eg \qquad (4)$$

Multiplying (1) and (2), and cancelling, we have $de > cf$. Similarly, multiplying (3) and (4), and cancelling, we get $cf > de$, contradicting $de > cf$.

Proposition 3.6

Suppose

$$B = \begin{pmatrix} B_1 & x \\ x^T & y \end{pmatrix}$$

is a real symmetric idempotent matrix, where x is a nonzero column. Then B_1 has exactly one eigenvalue α different from 0 and 1, and $\alpha(1-\alpha) = x^T x$. Furthermore, y = $1-\alpha$, and rank B_1= rank B.

Proof:

Since B_1 is real symmetric, there exists an orthogonal matrix Q_1 that diagonalizes B_1, that is,

$$Q_1^T B_1 Q_1 = D_1 = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Then $$\begin{pmatrix} Q_1^T & \\ & 1 \end{pmatrix} B \begin{pmatrix} Q_1 & \\ & 1 \end{pmatrix} = \begin{pmatrix} Q_1 & Q_1^T x \\ x^T Q_1 & y \end{pmatrix},$$

which is also a real symmetric idempotent matrix.

So, $D_1 = D_1^2 + Q_1^T x x^T Q_1$, or

$$\begin{pmatrix} \lambda_1 - \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_1 - \lambda_1^2 \end{pmatrix} = Q_1^T x x^T Q_1$$

The right hand side of this last equation has rank 1, and hence so does the left hand side. Hence, there is exactly one λ_j such that $\lambda_j - \lambda_j^2 \neq 0$ and $\lambda_j - \lambda_j^2 = tr(Q_1^T x^T x Q_1) = x^T x$. Let α denote this eigenvalue of B_1. We then have $\alpha - \alpha^2 = x^T x > 0$,so that 0< α< 1. Comparsion of the (2 , 2) blocks of B and B^2 gives $y - y^2 = x^T x$. Since both α and y are solutions to the equation $t - t^2 = x^T x$, it follows that $y = \alpha$ or 1- α.

We claim that $y = 1 - \alpha$. If $\alpha = 1 - \alpha$, we are done. Otherwise $\alpha \neq \frac{1}{2}$,$0 < \alpha < 1$. If $y = \alpha$, then

$$rank\ B = tr(B) = tr(B_1) + y = k + \alpha + \alpha,$$

Where k is the algebraic multiplicity of 1 as an eigenvalue of B_1. Since 2α is not an integer, we have a contradiction. Thus $y = 1 - \alpha$. It is now clear that

$$rank\ B_1 = 1 + (algebraic\ multiplicity\ of\ 1\ as\ an\ eigenvalue\ of B_1)$$

$$= 1 + (tr(B_1) - \alpha)$$

$$= tr(B_1) + (1 - \alpha)$$

$$= tr(B)$$

= rank B

4. Sign Patterns In ID or SID of orders ≤ 5

The 2×2 sign pattern in ID and SID are given. We are consider 3×3 are patterns in ID and patterns of orders 4 in SID.

Since the patterns of nonnegative idempotent matrices are known, it suffices to consider 3×3 sign patterns that are not signature similar to nonnegative patterns.

Proposition 4.1

Up to permutation similarity, signature similarity, and transposition, there are thirty three 3×3 sign patterns A such that $A^2 \overset{c}{\leftrightarrow} A$, A has at least one "+" diagonal entry, and A is not signature similar to a nonnegative pattern. Out of these 33 patterns, 13 are in ID, and 1 is in SID.

We obtain the 33 sign patterns mentioned above by using several Matlab programs. In turn, for each of the 33 patterns, we either produce an idempotent matrix (using some basic observations and Maple) or show that the patterns does not allow idempotence. The following idempotent matrices represent the 13 sign patterns in ID(the last one represents the one pattern in SID):

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1/2 & -1/2 \\ -1 & -1/2 & 1/2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 3 & 3 \\ -4/3 & -1 & -2 \\ -2/3 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 2 & 2 \\ -3/4 & -1/2 & -3/2 \\ -1/4 & -1/2 & 1/2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 2 \\ -1/2 & 0 & -1 \\ -1/2 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 3/2 & 3/2 & 3/2 \\ -1/3 & 0 & -1 \\ -1/6 & -1/2 & 1/2 \end{pmatrix}, \begin{pmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{pmatrix}.$$

In constructing several of the above matrices, the following two facts were used:

(i) $\begin{pmatrix} I & 0 \\ u & B \end{pmatrix}$ is idempotent iff B is idempotent and $B_u = 0$

(ii) $\begin{pmatrix} B & 0 \\ v & 0 \end{pmatrix}$ is idempotent iff B is idempotent and $(B^T - I)v^T = 0.$

As of the 20 patterns not in *ID* mentioned in the above results. Consider

$$A = \begin{pmatrix} + & + & + \\ - & - & - \\ + & + & + \end{pmatrix}$$

Suppose $B \in Q(A)$ is idempotent. Then it can be seen that rank B = 2 = tr(B). Therefore, rank (B – I) = rank (I – B) = 1, so that we must have

$$\text{sgn}\,(B - I) = \begin{pmatrix} + & + & + \\ - & - & - \\ + & + & + \end{pmatrix}$$

Hence, $b_{11} > 1, b_{33} > 1.$ since $b_{22} > 0$, we then get tr(B) > 2, a contradiction.

Proposition 4.2

Up to permutation similarity and signature similarity, there are five 4×4 irreducible symmetric sign patterns A such that $A^2 \overset{c}{\leftrightarrow} A$. All of these 5 patterns are in SID.

As in the 3×3 case, we obtain the 5 patterns mentioned above by using several Matlab programs. we find the following idempotent matrices representing the 5

sign patterns in SID:

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \frac{1}{4}\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix} 4-s & 1 & 1 & 0 \\ 1 & s & 0 & -1 \\ 1 & 0 & s & 1 \\ 0 & -1 & 1 & 4-s \end{pmatrix}, \quad \text{where } s = 2 \pm \sqrt{2},$$

$$\frac{1}{4}\begin{pmatrix} 2+\sqrt{2} & 1 & 1 & 0 \\ 1 & 3-\sqrt{2} & -1 & -t \\ 1 & -1 & 3-\sqrt{2} & t \\ 0 & -t & t & \sqrt{2} \end{pmatrix}, \quad \text{where } t = \sqrt{2\sqrt{2}-1}$$

5. Conclusion

Hence the complete characterizations of the classes ID and SID still remain open, as well as a number of other open questions involving these classes. For an $n \times n$ nonnegative pattern $A \in ID$, rank B = *mr*A for all idempotents $B \in Q(A)$, and furthermore, for each $1 \leq r \leq n$, there exists an $r \times r$ principal submatrix of A which is in ID. Let B be a real square matrix. Then, B is idempotent if and only if $(2B - I)^2 = I$ and B is symmetric idempotent if and only if 2B – I is a symmetric orthogonal matrix. Hence, B is idempotent $\Rightarrow$ sgn (2B –I) allows an inverse pair and B is symmetric idempotent ⇒ sgn (2B - I) allows a symmetric orthogonal matrix. sgn B and sgn (2B – I) can differ only on the diagonal. The patterns that allow an orthogonal matrix have recently been investigated.

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