Starlike and Convex Functions Subordinate to Leaf-Like Domain

Gurmeet Singh[1], Chatinder Kaur[2]
[1]Department of Mathematics, GSSDGS Khalsa College Patiala
[2]Research fellow, Department of Mathematics, Punjabi University, Patiala.
Email: meetgur111@gmail.com[1], chatinderkaur@gmail.com[2]

ABSTRACT: The aim of present paper is to find the Fekete-Szego inequality for the Starlike and Convex functions associated with Leaf-Like domains. Furthermore, similar study has done for convolution of functions.

Introduction:

Let A denote the class of functions f having the Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the region $E = \{z; |z| < 1\}$. We have subfamily G of A, which consists univalent functions.

The work of Raina and Sokol [2], Sokol ans Thomas [3] and Hari Priya [1] motivates us to introduce the function $l(z) = z + (1 + z^3)^{\frac{1}{3}}$ which maps the unit disc onto analytic and univalent region which has the shape of a leaf-like. This function has a symmetry with respect to the real axis. Real part of this function is positivewith conditions $l(0) = l'(0) = 1$. Following Lemma plays an important role to prove our results

Lemma1: Let $P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ be an analytic function in the region E with the property $P(0) = 1$ then

$$|c_n| \leq 2 \text{ for all } n \geq 1 \text{ and } \left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

P is the class of all such functions which has the property of positive real part.

Lemma2: Let the analytic function $P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$, which have positive real part, then we have

$$\left| c_2 - \mu c_1^2 \right| \leq 2 \max\{1, |2\mu - 1|\}$$

here μ is the complex number

Functions $P(z) = \frac{1+z^2}{1-z^2}$ and $P(z) = \frac{1+z}{1-z}$ provides the sharp result.

Fekete Szego coefficient function for the function f in the class S^*.

Theorem 1: If $f \in S^*$ then

$$\left| a_3 - \mu a_2^2 \right| \leq \frac{1}{2} \max\{1, |2\mu - 1|\}$$

And the result is sharp.

Proof. If $f \in S^*$ then for the Schwartz function w with $w(0)=0$ and $|w(z)| \leq 1$ we have

$$\frac{zf'(z)}{f(z)} = w(z) + \sqrt[3]{1+(w(z))^3} \tag{1.1}$$

we have

$$P(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$
$$w(z) = \frac{P(z)-1}{P(z)+1} \tag{1.2}$$

From this , we have the right side of (1.1)

$$\frac{P(z)-1}{P(z)+1} + \sqrt[3]{1+\left\{\frac{P(z)-1}{P(z)+1}\right\}^3} = 1 + \frac{c_1}{2}z + \left\{\frac{c_2}{2} - \frac{c_1^2}{4}\right\}z^2 + \left\{\frac{c_3}{2} - \frac{c_1 c_2}{2} + \frac{c_1^3}{6}\right\}z^3 + \cdots \tag{1.3}$$

Now as we have the function

$$\frac{zf'(z)}{f(z)} = \frac{1+2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots}{1 + a_2 z + a_3 z^2 + a_4 z^3 + a_5 z^4 + \cdots} \tag{1.4}$$

Now from (1.1), (1.3) and (1.4) , we have

$$1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots =$$
$$\left(1 + a_2 z + a_3 z^2 + a_4 z^3 + \cdots\right)\left(1 + \frac{c_1}{2}z + \left\{\frac{c_2}{2} - \frac{c_1^2}{4}\right\}z^2 + \left\{\frac{c_3}{2} - \frac{c_1 c_2}{2} + \frac{c_1^3}{6}\right\}z^3 + \cdots\right)$$

Equating coefficients of z , we get

$$a_2 = \frac{c_1}{2} \quad \text{and} \quad a_3 = \frac{c_2}{4}$$

So we have

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{4}\left|c_2 - \mu c_1^2\right| \tag{1.5}$$

Now applying lemma 2 on the equation (1.5) , then we have

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2}\max\{1, |2\mu - 1|\}$$

So we have

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{1}{2}, & p(z) = \frac{1+z^2}{1-z^2} \\ \frac{1}{2}|2\mu - 1|, & p(z) = \frac{1+z}{1-z} \end{cases} \tag{1.6}$$

The result is sharp.

Fekete Szego coefficient function for the function f in the class K.

Theorem 2: If $f \in K$ then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{6}\max\left\{1, \left|\frac{3}{2}\mu - 1\right|\right\}$$

And the result is sharp.

Proof. If $f \in K$ then for the Schwartz function w with $w(0)=0$ and $|w(z)| \leq 1$ we have

$$\frac{(zf'(z))'}{f'(z)} = w(z) + \sqrt[3]{1+(w(z))^3} \tag{2.1}$$

we have

$$P(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$
$$w(z) = \frac{P(z)-1}{P(z)+1} \tag{2.2}$$

From this , we have the right side of (2.1)

$$\frac{P(z)-1}{P(z)+1} + \sqrt[3]{1+\left\{\frac{P(z)-1}{P(z)+1}\right\}^3} = 1 + \frac{c_1}{2}z + \left\{\frac{c_2}{2} - \frac{c_1^2}{4}\right\}z^2 + \left\{\frac{c_3}{2} - \frac{c_1c_2}{2} + \frac{c_1^3}{6}\right\}z^3 + \cdots \tag{2.3}$$

Now as we have the function

$$\frac{(zf'(z))}{f'(z)} = \frac{1+4a_2z+9a_3z^2+16a_4z^3+25a_5z^4+\cdots}{1+2a_2z+3a_3z^2+4a_4z^3+5a_5z^4+\cdots} \tag{2.4}$$

Now from (2.1), (2.3) and (2.4) , we have

$$1+4a_2z+9a_3z^2+16a_4z^3+\cdots=$$
$$\left(1+2a_2z+3a_3z^2+4a_4z^3+\cdots\right)\left(1+\frac{c_1}{2}z+\left\{\frac{c_2}{2}-\frac{c_1^2}{4}\right\}z^2+\left\{\frac{c_3}{2}-\frac{c_1c_2}{2}+\frac{c_1^3}{6}\right\}z^3+\cdots\right)$$

Equating coefficients of z , we get

$$a_2 = \frac{c_1}{4} \quad \text{and} \quad a_3 = \frac{c_2}{12}$$

So we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{12}\left|c_2 - \frac{3}{4}\mu c_1^2\right| \tag{2.5}$$

Now applying lemma 2 on the equation (2.5) , then we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{6}\max\left\{1, \left|\frac{3}{2}\mu - 1\right|\right\}$$

So we have

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{1}{6}, & p(z) = \frac{1+z^2}{1-z^2} \\ \frac{1}{6}\left|\frac{3}{2}\mu - 1\right|, & p(z) = \frac{1+z}{1-z} \end{cases} \tag{2.6}$$

The result is sharp.

Theorem 3:Fekete Szegö coefficient function for the fuction f in the class $f \in S^*(f*g)$.

If $f \in S*(f*g)$then

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{2b_3}\max\left\{1, \left|\frac{2b_3}{b_2}\mu - 1\right|\right\}$$

And the result is sharp.

Proof. If $f \in S^*(f * g)$ then for the Schwartz function w with $w(0)=0$ and $|w(z)| \le 1$ we have

$$\frac{z(f * g)'(z)}{(f * g)(z)} = w(z) + \sqrt[3]{1+(w(z))^3} \tag{3.1}$$

we have

$$\begin{aligned} P(z) &= \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \\ w(z) &= \frac{P(z)-1}{P(z)+1} \end{aligned} \tag{3.2}$$

From this , we have the right side of (1.1)

$$\frac{P(z)-1}{P(z)+1} + \sqrt[3]{1+\left\{\frac{P(z)-1}{P(z)+1}\right\}^3} = 1 + \frac{c_1}{2} z + \left\{\frac{c_2}{2} - \frac{c_1^2}{4}\right\} z^2 + \left\{\frac{c_3}{2} - \frac{c_1 c_2}{2} + \frac{c_1^3}{6}\right\} z^3 + \cdots \tag{3.3}$$

Now as we have the function

$$\frac{z(f * g)'(z)}{(f * g)(z)} = \frac{1 + 2a_2 b_2 z + 3a_3 b_3 z^2 + 4a_4 b_4 z^3 + 5a_5 b_5 z^4 + \cdots}{1 + a_2 b_2 z + a_3 b_3 z^2 + a_4 b_4 z^3 + a_5 b_5 z^4 + \cdots} \tag{3.4}$$

Now from (3.1), (3.3) and (3.4) , we have

$$1 + 2a_2 b_2 z + 3a_3 b_3 z^2 + 4a_4 b_4 z^3 + \cdots =$$

$$\left(1 + a_2 b_2 z + a_3 b_3 z^2 + a_4 b_4 z^3 + \cdots\right)\left(1 + \frac{c_1}{2} z + \left\{\frac{c_2}{2} - \frac{c_1^2}{4}\right\} z^2 + \left\{\frac{c_3}{2} - \frac{c_1 c_2}{2} + \frac{c_1^3}{6}\right\} z^3 + \cdots\right)$$

Equating coefficients of z, we get

$$a_2 = \frac{c_1}{2b_2} \quad \text{and} \quad a_3 = \frac{c_2}{4b_3}$$

So we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{4b_3}\left|c_2 - \frac{\mu c_1^2 b_3}{b_3^2}\right| \tag{3.5}$$

Now applying lemma 2 on the equation (3.5) , then we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{2} \max\left\{\frac{1}{b_3}, \frac{1}{b_3}\left|\frac{2\mu b_3}{b_2} - 1\right|\right\}$$

So we have

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{1}{2b_3}, & p(z) = \dfrac{1+z^2}{1-z^2} \\ \dfrac{1}{2b_3}\left|\dfrac{2\mu b_3}{b_2} - 1\right|, & p(z) = \dfrac{1+z}{1-z} \end{cases} \tag{3.6}$$

The result is sharp.

References

[1]Hari, P. M. and Sharma, R.B., 2018, April. *On a class of bounded turning functions subordinate to a leaf-like domain.* In Journal of Physics: Conference Series 1000, (1), p. 012056.

[2] Raina R and Sokol J 2015 *On Coefficient estimates for a certain class of starlike functions*, Hacettepe Journal of Mathematics and Statistics 44(6) ,pp.1427-1433.

[3]Dziok, J., Raina, R.K. and Sokół, J., 2011. *On α-convex functions related to shell-like functions connected with Fibonacci numbers*. Applied Mathematics and Computation, 218(3), pp.996-1002.

[4]Duren P.L. (1977), *Coefficient of univalent functions*, Bulletin of American Mathematical Society, 83, pp.891-911

[5] Fekete M. and Szeg G., ,8(1933): Eine bemerkung ber ungerade schlichte funktionen, J London Math. Soc., pp.85-89.

[6]Nehari Z. (1952), *Conformal Mappings*, Bulletin of the American Mathematical Society 58.4, pp.515-517.