Study of Heat Generation and Chemical Reaction of spilled oil in the Subsurface with Variable Porosity

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Abstract

An analysis is presented for the effects of heat generation and chemical reaction with porosity variation of spilled oil in the subsurface. The dimensionless governing equations are solved using the perturbation method. The equations of momentum includes thermal and concentration buoyancy effects, energy equation include heat generation and the equation of species includes in chemical reaction. Solution of velocity, temperature and concentration profiles are presented graphically with several values of parameter.

Keywords: Heat Generation, Chemical Reaction, Porosity Variation, Perturbation technique.

1 Introduction

In certain application such as those dealing with heat absorption/generation and chemical reaction effects may alter the thermal and concentration distribution. It is play a vital role in many respects for instance in flows in soil, solar power collectors, heat transfer in aerobic and anaerobic reaction etc. The numerical analysis of chemical reaction effects on heat and mass flow of a permeable channel with radiation and heat generation was presented by Singh and Kumar (2016). The effects of heat generation and chemical reaction on magnetohydrodynamic convection flow over a porous medium is studied by Khan et al., (2019), Chamkha et al., (2006) and Suresh et al., (2019). Effects of hydrocarbon generation, basal heat flow and sediment compaction on overpressure is analyzed by Hansom and Kuo Lee (2005). Vidonish et al., (2016) discussed thermal treatment of hydrocarbon impacted soils. Lavanya and Nagasasikala (2016) analyse the influence of the convective flow of a viscous incompressible fluid through a porous medium with vertical infinite moving plate in the presence of heat generation and soret effect.

The study of heat generation with porosity have been developed by several researcher. The effects of porosity of the medium and heat generation on the isotherms was conducted by Mousa (2019). Skibinski et al., (2019) studied the impact of pore size variation on the effective thermal conductivity on porous foams. Basu et al., (2019) discussed temperature variation is investigated for a range of values absolute permeability and porosity of the porous medium.

In this problem, the effect of heat absorption and chemical reaction of spilled oil in the subsurface with variable porosity is analyzed. Firstly, the governing equation have been made non dimensional and solved semi analytically. The flow equation includes temperature and concentration buoyancy effects. The heat transfer occurs mostly in nature due to thermal gradient. The species equation of concentration includes chemical reaction.

2 Mathematical formulation

consider the unsteady, laminar and two dimensional geometry with porosity variation in the horizontal channel. The graphical model of the problem are illustrated in figure 1. The equation of momentum in the presence of thermal

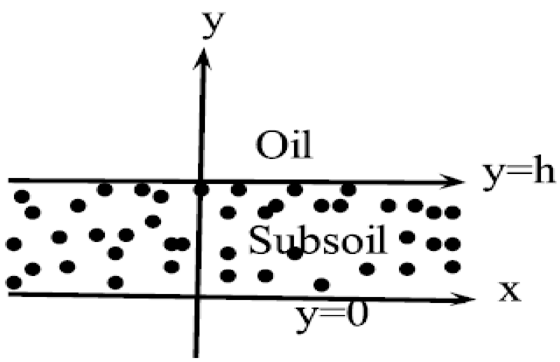

Figure 1: Physical configuration.

and concentration buoyancy effects including Darcy term.

Based on these assumptions, the governing equations are given by.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{1}{\Theta}\frac{\partial u}{\partial t} + \frac{u}{\Theta}\frac{\partial}{\partial x}(\frac{u}{\Theta}) + \frac{v}{\Theta}\frac{\partial}{\partial y}(\frac{u}{\Theta}) = \frac{\nu}{\Theta}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

$$+g\beta_T(T - T_1) + g\beta_c(c - c_1) - \frac{\nu}{k_p}u \tag{2}$$

$$\frac{1}{\Theta}\frac{\partial v}{\partial t} + \frac{u}{\Theta}\frac{\partial}{\partial x}(\frac{v}{\Theta}) + \frac{v}{\Theta}\frac{\partial}{\partial y}(\frac{v}{\Theta}) = \frac{\nu}{\Theta}(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) - \frac{\nu}{k_p}v \tag{3}$$

$$\Theta\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \Theta\frac{k_T}{\rho c_p}(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) + \frac{Q_0}{\rho c_p}(T - T_1) \tag{4}$$

$$\rho_b\frac{\partial s}{\partial t}+\Theta\beta_w\frac{\partial c}{\partial t}+\beta_w u\frac{\partial c}{\partial x}+\beta_w v\frac{\partial c}{\partial y}=\Theta\beta_w D_m(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial y^2})$$

$$-\beta_w k_1(c-c_1) \tag{5}$$

Let u, v be the velocities of the fluid particle along x and y axes, T and c denotes the thermal and concentration distribution, s denotes the concentration of adsorbed oil in the subsurface, t denotes the time, ρ denotes the density of fluid, ν represents kinematic viscosity, ρ_b denotes the soil bulk density, β_w represents the volumetric water content in the soil, D_m denotes the diffusivity of mass, T_1 and c_1 denotes the thermal and concentration at the upper surface, k_p denotes the permeability of the medium, k_1 denotes the chemical reaction parameter and Θ is the porosity. Nirmala Ratchagar and senthamilselvi (2019) analyzed variable porosity can be expressed as $\Theta=\Theta_s a_1 e^{\frac{-a_2 y}{d_p}}$ where Θ_s is the mean porosity, a_1, a_2 are empirical constants and d_p is the particle diameter. The retardation factor defined as $R=1+\frac{\rho_b k_d}{\beta_w}$, where $s=k_d\ c$ and k_d is the adsorption coefficient. This reduces the equation (5) is

$$\Theta R\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y}=\Theta D_m(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial y^2})-k_1(c-c_1) \tag{6}$$

The initial and boundary conditions for the flow are,

$$u=0,\ v=0,\ T=T_0,\ c=c_0\ \forall y,\ t\le 0 \tag{7}$$

$$\frac{\partial u}{\partial y}=\frac{-\alpha}{\sqrt{k_p}}u,\ v=0,\ T=T_0\epsilon e^{i(\omega t+\lambda x)}(T_0-T_1),$$

$$c=c_0\epsilon e^{i(\omega t+\lambda x)}(c_0-c_1)\ at\ y=0,\ t>0 \tag{8}$$

$$\frac{\partial u}{\partial y}=\frac{\alpha}{\sqrt{k_p}}u,\ v=-\epsilon e^{i(\omega t+\lambda x)}v_0,\ T=T_1,\ c=c_1\ at\ y=h,\ t>0 \tag{9}$$

where, α represents the slip parameter, T_0 and c_0 be the initial thermal and concentration of hydrocarbon, λ represents the stream wise wave number, ω represents the frequency parameter, ϵ represents the perturbation parameter and i represents the imaginary part.

To write the flow model in dimensionless quantities, we use the following parameters

$x^* = \frac{xv_0}{\nu}, y^* = \frac{yv_0}{\nu}, u^* = \frac{u}{v_0}, v^* = \frac{v}{v_0}, t^* = \frac{tv_0}{\nu},$

$T^* = \frac{T-T_1}{T_0-T_1},\ \phi = \frac{c-c1}{c_0-c1}, dp^* = \frac{dpv_0}{\nu}$

where, ϕ is the non dimensional variables in concentration, v_0 is the characteristic velocity. Applying the dimensionless variables from the equations (1) to (4) and (6) omitting the asterisk symbol we obtained,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{10}$$

$$\frac{1}{\Theta}\frac{\partial u}{\partial t} + \frac{u}{\Theta^2}\frac{\partial u}{\partial x} + \frac{v}{\Theta^3}(\frac{\Theta\partial u}{\partial y} - u\frac{\partial\Theta}{\partial y}) = \frac{1}{\Theta}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + GrT + Gc\phi - \sigma^2 u \tag{11}$$

$$\frac{1}{\Theta}\frac{\partial v}{\partial t} + \frac{u}{\Theta^2}\frac{\partial v}{\partial x} + \frac{v}{\Theta^3}(\frac{\Theta\partial v}{\partial y} - v\frac{\partial\Theta}{\partial y}) = \frac{1}{\Theta}(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) - \sigma^2 v \tag{12}$$

$$\Theta\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\Theta}{Pr}(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) + Q_1 T \tag{13}$$

$$\Theta R\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = \frac{\Theta}{Sc}(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}) - K\phi \tag{14}$$

where, $Gr = \frac{\nu g\beta_T(T_0-T_1)}{v_0^3}$ represents the Grashof number, $Gc = \frac{\nu g\beta_c(c_0-c_1)}{v_0^3}$ represents the modified Grashof number, $\sigma = \frac{\nu}{v_0\sqrt{k_p}}$ represents the porous parameter, $Pr = \frac{\rho c_p\nu}{k_T}$ represents the Prandtl number, $Q_1 = \frac{Q_0\nu}{\rho c_p v_0^2}$ represents the heat generation parmeter, $Sc = \frac{\nu}{D_m}$ represents the Schmidt number and $K = \frac{k_1\nu}{v_0^2}$ represents the chemical reaction rate parameter.

The non dimensional form initial and boundary conditions are:

$$u = 0,\ v = 0,\ T = 1,\ \phi = 1\ \ \forall y,\ t \leq 0 \tag{15}$$

$$\frac{\partial u}{\partial y} = -\alpha\sigma u,\ v = 0,\ T = 1 + \epsilon e^{i(\omega t + \lambda x)},$$

$$\phi = 1 + \epsilon e^{i(\omega t + \lambda x)}\ at\ y = 0,\ t > 0 \tag{16}$$

$$\frac{\partial u}{\partial y} = \alpha\sigma u,\ v = -\epsilon e^{i(\omega t + \lambda x)},\ T = 0,\ \phi = 0\ at\ y = 1,\ t > 0 \tag{17}$$

3 Method of solution

The system of equation (10) to (17). We decompose the flow variables into base and perturbed part as follows.

$$\left.\begin{aligned} u(x,y,t) &= u_0(y) + \epsilon e^{i(\omega t + \lambda x)}\overline{u_1(y)} + o(\epsilon^2) \\ v(x,y,t) &= \epsilon e^{i(\omega t + \lambda x)}\overline{v_1(y)} + o(\epsilon^2) \\ T(x,y,t) &= T_0(y) + \epsilon e^{i(\omega t + \lambda x)}\overline{T_1(y)} + o(\epsilon^2) \\ \phi(x,y,t) &= \phi_0(y) + \epsilon e^{i(\omega t + \lambda x)}\overline{\phi_1(y)} + o(\epsilon^2) \end{aligned}\right\} \tag{18}$$

Substituting the equation (18) into the equations (10) to (14) neglecting the higher order of perturbation parameter (ϵ^2) and equating the base and perturbed part.

Base part:

$$\frac{\partial^2 u_0}{\partial y^2} - \Theta\sigma^2 u_0 + \Theta Gr T_0 + \Theta Gc\phi_0 = 0 \tag{19}$$

$$\frac{\partial^2 T_0}{\partial y^2} + \frac{Pr Q_1}{\Theta} T_0 = 0 \tag{20}$$

$$\frac{\partial^2 \phi_0}{\partial u^2} - \frac{Sc}{\Theta} K\phi_0 = 0 \tag{21}$$

with the boundary conditions:

$$\frac{\partial u_0}{\partial y} = -\alpha\sigma u_0,\ T_0 = 1,\ \phi_0 = 1\ at\ y = 0,\ t > 0 \tag{22}$$

$$\frac{\partial u_0}{\partial y} = \alpha\sigma u_0,\ T_0 = 0,\ \phi_0 = 0\ at\ y = 1,\ t > 0 \tag{23}$$

The equations (19), (20) and (21) are transforming by "change the independent variable methods".

We get the solutions are.

$$u_0 = e^{\frac{2}{A_2}\sqrt{A_1\sigma^2 e^{A2y}}}L_1 + e^{-\frac{2}{A_2}\sqrt{A_1\sigma^2 e^{A2y}}}L_2 + \frac{Gr}{\sigma^2}(L_3Cos[\frac{2\sqrt{f_2e^{-A_2y}}}{(A_2)^{3/2}}]$$

$$-L_4Sin[\frac{2\sqrt{f_2e^{-A_2y}}}{(A_2)^{3/2}}]) + \frac{Gc}{\sigma^2}(L_5e^{\frac{2}{A_2}\sqrt{\frac{f_1}{A_1}e^{-A_2y}}} + L_6e^{-\frac{2}{A_2}\sqrt{\frac{f_1}{A_1}e^{-A_2y}}}) \tag{24}$$

$$T_0 = L_3Cos[\frac{2\sqrt{f_2e^{-A_2y}}}{(A_2)^{3/2}}] - L_4Sin[\frac{2\sqrt{f_2e^{-A_2y}}}{(A_2)^{3/2}}] \tag{25}$$

$$\phi_0 = L_5e^{\frac{2}{A_2}\sqrt{\frac{f_1}{A_1}e^{-A_2y}}} + L_6e^{-\frac{2}{A_2}\sqrt{\frac{f_1}{A_1}e^{-A_2y}}} \tag{26}$$

where the constants L_i for i=1 *to* 8 are given in the appendix.

Perturbed part:

$$\frac{\partial^2 u_1}{\partial y^2} + [(\omega + \frac{\lambda u_0}{\Theta})tan(\lambda x + \omega t) - \Theta\sigma^2 - \lambda^2]u_1 +$$

$$\frac{v_1}{\Theta}[\frac{1}{\Theta}u_0\frac{\partial\Theta}{\partial y} - \frac{\partial u_0}{\partial y}] + \Theta Gr T_1 + \Theta Gc\phi_1 = 0 \tag{27}$$

$$\frac{\partial^2 v_1}{\partial y^2} + [(\omega + \frac{\lambda u_0}{\Theta})tan(\lambda x + \omega t) - \Theta\sigma^2 - \lambda^2]v_1 = 0 \tag{28}$$

$$\frac{\partial^2 T_1}{\partial y^2} + [Pr(\omega + \frac{\lambda u_0}{\Theta})tan(\lambda x + \omega t) - \lambda^2 + \frac{PrQ_1}{\Theta}]T_1 - \frac{Pr}{\Theta}v_1\frac{\partial T_0}{\partial y} = 0 \quad (29)$$

$$\frac{\partial^2 \phi_1}{\partial y^2} + [Sc(R\omega + \frac{\lambda u_0}{\Theta})tan(\lambda x + \omega t) - \lambda^2 - \frac{Sck}{\Theta}]\phi_1 - \frac{Sc}{\Theta}v_1\frac{\partial \phi_0}{\partial y} = 0 \quad (30)$$

With the boundary condition:

$$\frac{\partial u_1}{\partial y} = -\alpha\sigma u_1, v_1 = 0, T_1 = 1, \phi_1 = 1 \ at \ y = 0 \quad (31)$$

$$\frac{\partial u_1}{\partial y} = \alpha\sigma u_1, v_1 = -1, T_1 = 0, \phi_1 = 0 \ at \ y = 1 \quad (32)$$

4 Results and Discussion

The problem concerned with the effects of heat generation and chemical reaction of spilled oil in the subsurface with variable porosity has been discussed. The expression for velocity, temperature and concentration which have been computed numerically using mathematica software and results are analyzed graphically.

For several values of porous (σ) and Grashof number (Gr) on the velocity (u) is plotted in figures 2 and 3. Both figures we see that improving the parameters values promotes the velocity field. Figure 4 depicts the velocity (u) for various values of particle diameter dp. We see that decreasing velocity distribution with the increasing values of the parameter. In the case when the flow is vertical direction (v) is given in figure 5. It is observed that the velocity enhances with increase in porous parameter. As the porous parameter the frictional drag resistance against the flow in the porous region is very large and as, the velocity decelerate for accelerating porous parameter.

Figures 6, 7, 8 and 9 signify the Prandtl number (Pr), porous (σ),

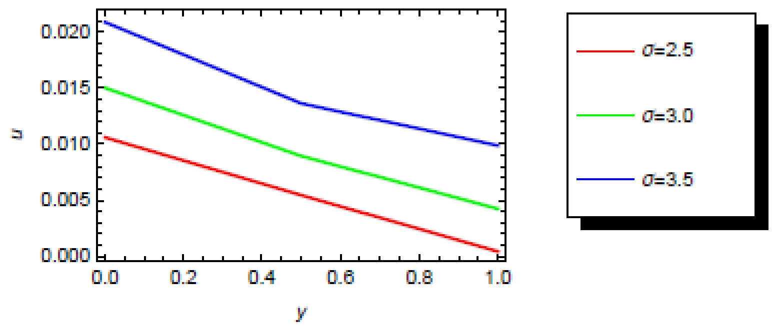

Figure 2: Effect of varying the porous parameter on u velocity.

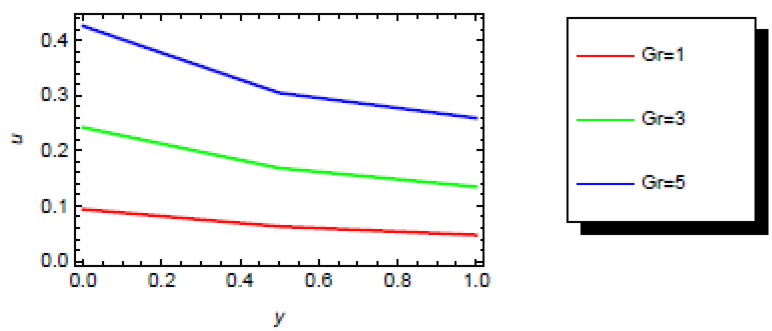

Figure 3: Effect of varying the Grashof number on u velocity.

heat generation Q and particle diameter dp on temperature distribution. Figure 6 indicates that temperature distribution is decrease with increasing the parameter. Figure 7 we see that when the porous parameter increases the temperature increases. Figures 8 and 9 reveal that the parameters promotes the temperature distribution. The feature of graphs change the parabolic. Figures 10, 11 and 12 reveals that the effect of Schmidt number, chemical reaction and retardation factor on the concentration profiles. Figures 10 and 11 shows the hydrocarbon concentration decrease due to a rise in parameter values. From figure 12 it is observed that concentration traveling at constant speed.

5 Conclusion

This study focused on the migration of oil flow in the subsurface with variable porosity. When the petroleum compounds such as hydrocarbon are released into the soil, the compounds undergo physical and chemical properties. In

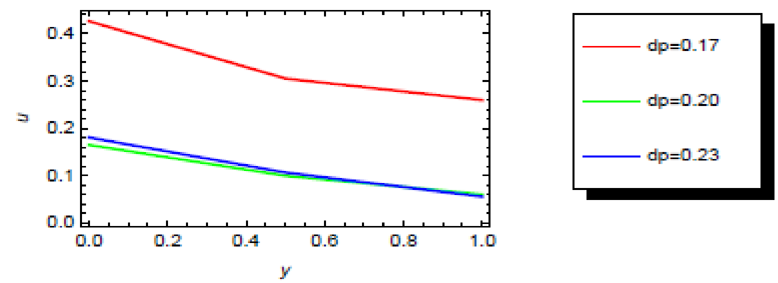

Figure 4: Effect of varying the particle diameter on u velocity.

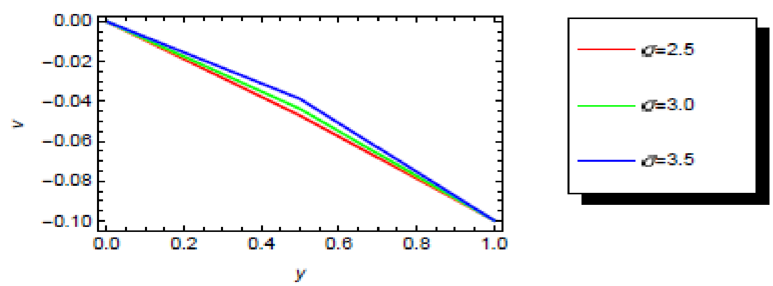

Figure 5: Effect of varying the porous parameter on v velocity.

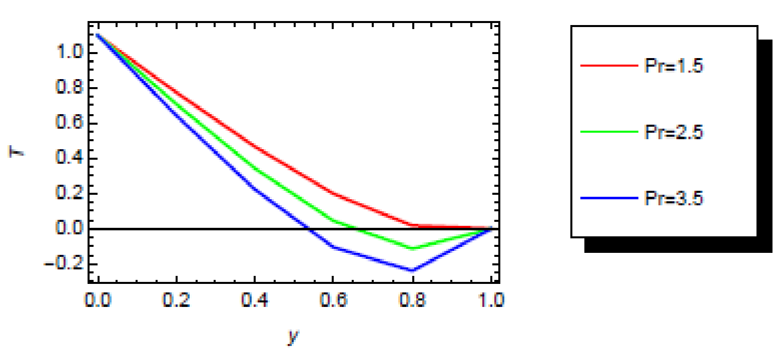

Figure 6: Effect of varying the Prandtl number on temperature distribution.

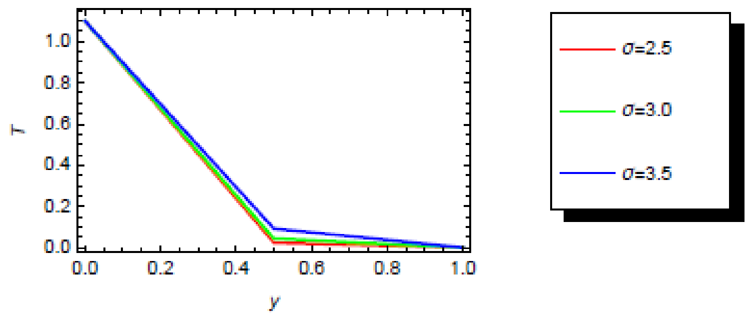

Figure 7: Effect of varying the porous parameter on temperature distribution.

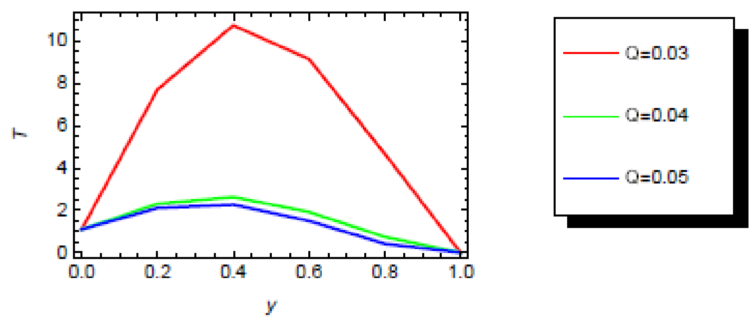

Figure 8: Effect of varying the heat generation parameter on temperature distribution.

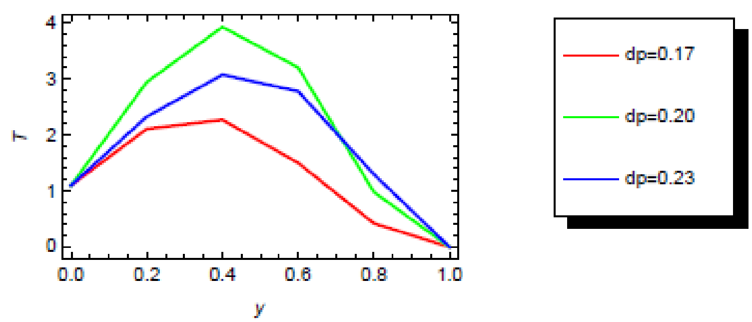

Figure 9: Effect of varying the particle diameter on temperature distribution.

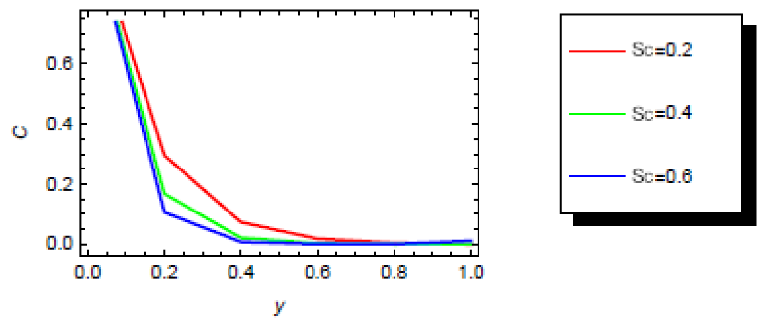

Figure 10: Effect of varying the Schmidt number on concentration distribution.

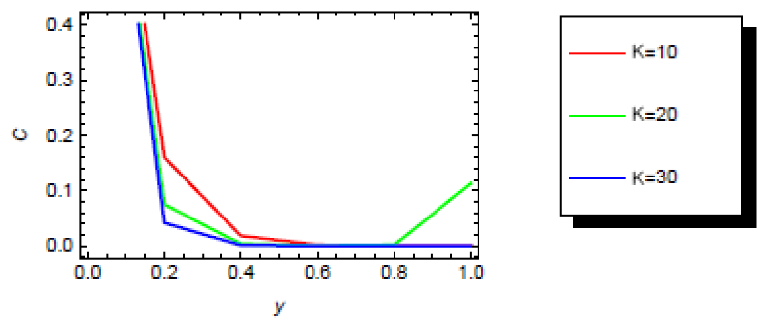

Figure 11: Effect of varying the chemical reaction parameter on concentration distribution.

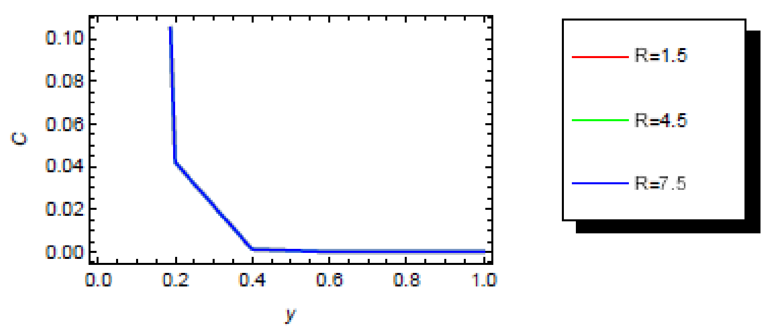

Figure 12: Effect of varying the retardation factor on concentration distribution.

conclusion , the analysis of the study show that soil characteristic play an important role in controlling the effect of hydrocarbon contaminants on the subsurface.

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Appendix

$$A_1 = \Theta_s a_1;$$

$$A_2 = \frac{-a_2}{dp};$$

$$f_1 = Sc\ k;$$

$$f_2 = Pr\ Q_1;$$

$$L_1 = \left[\frac{1}{\sqrt{A1\sigma^2}(A_1^2 e^{A_2}(L_7)\sigma^2 + (L_7)\alpha^2\sqrt{A_1\sigma^2}L_8 + A_1(L_7)\alpha\sigma(e^{A_2}\sqrt{A_1\sigma^2} + L_8))}\right]$$

$$-\Bigg[(A_1 e^{\frac{2(\sqrt{A_1\sigma^2}+L_8)}{A_2}} L_8(\frac{1}{\sqrt{A_2}f_1\sigma L_8} e^{-\frac{2(\sqrt{\frac{f_1}{A_1}}+L_8}{A_2}}$$

$$\sqrt{\frac{f_1}{A_1}}(A_1 e^{A_2}\sigma + \alpha L_8(-\sqrt{A_2}Gc(e^{\frac{4\sqrt{\frac{f_1}{A_1}}}{A_2}} L_5(f_1 - A_1\sqrt{\frac{f_1}{A_1}}\alpha\sigma)$$

$$-L_6(f_1 + A_1\sqrt{\frac{f_1}{A_1}}\alpha\sigma)) + A_1 e^{\frac{2\sqrt{\frac{f1}{A1}}}{A2}}\sqrt{\frac{f_1}{A_1}}Gr(\sqrt{f_2}L_4 + \sqrt{A_2}L_3\alpha\sigma)$$

$$Cos[\frac{2\sqrt{f_2}}{A_2^{\frac{3}{2}}}] + A_1 e^{\frac{2\sqrt{\frac{f_1}{A_1}}}{A_2}}\sqrt{\frac{f1}{A1}}Gr(\sqrt{f_2}L_3 - \sqrt{A_2}L_4\alpha\sigma)Sin[\frac{2\sqrt{f_2}}{A_2^{3/2}}])+$$

$$\frac{1}{\sqrt{A_2}(A_1\sigma^2)^{\frac{3}{2}}} A_1 e^{-\frac{2((\sqrt{\frac{e^{-A_2}f_1}{A_1}}+\sqrt{A_1\sigma^2})}{A_2}}\sigma(A_1\sigma - \alpha\sqrt{A_1\sigma^2})(\sqrt{A_2}Gc(-\sqrt{\frac{e^{-A_2}f_1}{A_1}}$$

$$L_6 + L_6\alpha\sigma + e^{\frac{4\sqrt{\frac{e^{-A_2}f_1}{A_1}}}{A_2}} L_5(\sqrt{\frac{e^{-A_2}f_1}{A_1}} + \alpha\sigma)) - e^{\frac{2\sqrt{\frac{e^{-A_2}f_1}{A_1}}}{A_2}} Gr(\sqrt{e^{-A_2}f_2}L_4$$

$$-\sqrt{A_2}L_3\alpha\sigma)Cos[\frac{2\sqrt{e^{-}A_2 f_2}}{A_2^{\frac{3}{2}}}] - e^{\frac{2\sqrt{\frac{e^{-A_2}f_1}{A_1}}}{A_2}} Gr$$

$$(\sqrt{e^{-A_2}f_2}L_3 + \sqrt{A_2}L_4\alpha\sigma)Sin[2\frac{\sqrt{e^{-}A_2 f_2}}{A^{\frac{3}{2}}}]))\Bigg];$$

$$L_2 = \left[\frac{1}{\sqrt{A1\sigma^2}(A_1^2e^{A_2}(L_7)\sigma^2 + (L_7)\alpha^2\sqrt{A_1\sigma^2}L_8 + A_1(L_7)\alpha\sigma(e^{A_2}\sqrt{A_1\sigma^2} + L_8))}\right]$$

$$-\Bigg[(e^{-\frac{2(\sqrt{\frac{f_1}{A_1}}+L_9-\sqrt{A_1\sigma_1}L_9+L_8)}{A_2}}(-A_1\sqrt{A_2}e^{A_2+\frac{2(2\sqrt{\frac{f_1}{A_1}}+L_9+L_8)}{A2}}\sqrt{\frac{f_1}{A_1}}GcL_5\sigma\sqrt{\frac{A_1}{\sigma^2}}+$$

$$A_1\sqrt{A_2}e^{A_2+\frac{2(L_9+L_8)}{A_2}}\sqrt{\frac{f_1}{A_1}}GcL_6\sigma\sqrt{\frac{A_1}{\sigma^2}} + \sqrt{A_2}e^{A_2+\frac{2(2\sqrt{\frac{f_1}{A_1}}+L_9+L_8)}{A2}}$$

$$GcL_5\alpha(A1\sigma^2)^{\frac{3}{2}} + \sqrt{A_2}e^{A_2+\frac{2(L_9+L_8)}{A2}}GcL_6\alpha(A1\sigma^2)^{\frac{3}{2}} + A_1\sqrt{A_2}e^{\frac{2(2\sqrt{\frac{f_1}{A_1}}+2L_9+\sqrt{A_1\sigma^2})}{A2}}L_9GcL_5$$

$$\sigma L_8 - A_1\sqrt{A_2}e^{\frac{2(\sqrt{\frac{f_1}{A_1}}+\sqrt{A_1\sigma^2})}{A2}}L_9GcL_6\sigma L_8 + A_1\sqrt{A_2}e^{\frac{2(2\sqrt{\frac{f_1}{A_1}}+2L_9+\sqrt{A_1\sigma^2})}{A2}}GcL_5\sigma^2L_8+$$

$$A_1\sqrt{A_2}e^{\frac{2(2\sqrt{\frac{f_1}{A_1}}+\sqrt{A_1\sigma^2})}{A2}}GcL_6\alpha\sigma^2L_8 + \sqrt{A_2}e^{\frac{2(2\sqrt{\frac{f_1}{A_1}}+2L_9+L_8)}{A2}}\sqrt{\frac{f_1}{A_1}}$$

$$\sqrt{A_2}e^{\frac{2(2\sqrt{\frac{f_1}{A_1}}+2L_9+\sqrt{A_1\sigma^2})}{A2}}\sqrt{\frac{e^{-A_2}f_1}{A_1}}GcL_5\alpha\sqrt{A_1\sigma^2}L_8 - GcL_5\alpha\sqrt{A_1\sigma^2}L_8+$$

$$\sqrt{A_2}e^{\frac{2(L_9+L_8)}{A_2}}\sqrt{\frac{f_1}{A_1}}GcL_6\alpha\sqrt{\frac{A_1}{\sigma^2}}L_8 - \sqrt{A_2}e\frac{2(\sqrt{\frac{f_1}{A_1}} + \sqrt{A_1\sigma^2})}{A_2}$$

$$\sqrt{\frac{e^{-A_2}f_1}{A_1}}GcL_6\alpha\sqrt{A_1\sigma^2}L_8 + \sqrt{A_2}e^{(\frac{2(\sqrt{\frac{f_1}{A_1}}+2L_9+\sqrt{A_1\sigma^2})}{A_2}}\sqrt{\frac{f_1}{A_1}}$$

$$GcL_5\alpha\sqrt{A_1\sigma^2}L_8 + \sqrt{A_2}e^{\frac{2(\sqrt{\frac{f_1}{A_1}}+2L_9+\sqrt{A_1\sigma^2})}{A_2}}\sqrt{\frac{e^{-A_2}f_1}{A_1}}GcL_5\alpha\sqrt{A_1\sigma^2}L_8-$$

$$\sqrt{A_2}e^{\frac{2(L_9+L_8)}{A_2}}\sqrt{\frac{f_1}{A_1}}GcL_6\alpha\sqrt{A_1\sigma^2}L_8 - \sqrt{A_2}e^{\frac{2(\sqrt{\frac{f_1}{A_1}}+\sqrt{A_1\sigma^2})}{A_2}}\sqrt{\frac{e^{-A_2}f_1}{A_1}}$$

$$GcL_6\alpha\sqrt{A_1\sigma^2}L_8 + \sqrt{A2}e^{\frac{2(\sqrt{\frac{f_1}{A_1}}+2L_9+\sqrt{A_1\sigma^2})}{A_2}}GcL_5\alpha^2\sigma\sqrt{A1\sigma^2}L_8\sqrt{A_2}e^{\frac{2(2\sqrt{\frac{f_1}{A_1}}+L_9+L_8)}{A_2}}$$

$$GcL_5\alpha^2\sigma\sqrt{A_1\sigma^2}L_8+\sqrt{A_2}e^{\frac{2(\sqrt{\frac{f_1}{A_1}}+\sqrt{A_1\sigma^2})}{A_2}}GcL_6\alpha^2\sigma\sqrt{A_1\sigma^2}L_8-\sqrt{A_2}e^{\frac{2(L_9+L_8)}{A_2}}$$

$$GcL_6\alpha^2\sigma\sqrt{A_1\sigma^2}L_8+e^{\frac{2(\sqrt{\frac{f_1}{A_1}}+L_9+L_8)}{A_2}}Gr\sqrt{A_1\sigma^2}(\sqrt{f_2}L_4+\sqrt{A_2}L_3\alpha\sigma)(A_1e^{A_2}\sigma-\alpha L_8)$$

$$Cos[\frac{2\sqrt{f_2}}{A_2^{\frac{3}{2}}}]+e^{\frac{2(\sqrt{\frac{f_1}{A_1}}+L_9+\sqrt{A_1\sigma^2})}{A_2}}GrL_8(-\sqrt{e^{-A_2}f_2}L_4+\sqrt{A_2}L_3\alpha\sigma)$$

$$(A_1\sigma+\alpha\sqrt{A_1\sigma^2})Cos[\frac{2\sqrt{e^{-A_2}f_2}}{A_2{\frac{3}{2}}}]+A_1e^{A_2+\frac{2(\sqrt{\frac{f_1}{A_1}}+L_9+L_8)}{A_2}}\sqrt{f_2}GrL_3\sigma\sqrt{A_1\sigma^2}$$

$$Sin[\frac{2\sqrt{f_2}}{A_2^{\frac{3}{2}}}]-\sqrt{A_2}e^{A_2+\frac{2(\sqrt{\frac{f_1}{A_1}}+L_9+L_8)}{A_2}}GrL_4\alpha(A_1\sigma^2)^{\frac{3}{2}}Sin[\frac{2\sqrt{f_2}}{A_2^{\frac{3}{2}}}]-e^{\frac{2(\sqrt{\frac{f_1}{A_1}}+L_9+L_8)}{A_2}}$$

$$\sqrt{f_2}GrL_3\alpha\sqrt{A_1\sigma^2}L_8Sin[\frac{2\sqrt{f_2}}{A_2^{\frac{3}{2}}}]+\sqrt{A_2}e^{\frac{2(\sqrt{\frac{f_1}{A_1}}+L_9+L_8)}{A_2}}GrL_4\alpha^2\sigma\sqrt{A_1\sigma^2}$$

$$L_8Sin[\frac{2\sqrt{f_2}}{A_2^{\frac{3}{2}}}]-A_1e^{\frac{2(\sqrt{\frac{f_1}{A_1}}+L_9+\sqrt{A_1\sigma^2})}{A_2}}\sqrt{e^{-A_2}f_2}GrL_3\sigma L_8Sin[\frac{2\sqrt{e^{-A_2}f_2}}{A_2^{\frac{3}{2}}}]-$$

$$A_1\sqrt{A_2}e^{\frac{2(\sqrt{\frac{f_1}{A_1}}+L_9+\sqrt{A_1\sigma^2})}{A_2}}GrL_4\alpha\sigma^2L_8Sin[\frac{2\sqrt{e^{-A_2}f_2}}{A_2^{\frac{3}{2}}}]-e^{\frac{2(\sqrt{\frac{f_1}{A_1}}+L_9+\sqrt{A_1\sigma^2})}{A_2}}$$

$$\sqrt{e^{-A_2}f_2}GrL_3\alpha\sqrt{A_1\sigma^2}L_8Sin[\frac{2\sqrt{e^{-A_2}f_2}}{A2^{\frac{3}{2}}}]-\sqrt{A_2}e^{\frac{2(\sqrt{\frac{f_1}{A_1}}+L_9+\sqrt{A_1\sigma^2})}{A2}}$$

$$GrL_4\alpha^2\sigma\sqrt{A_1\sigma^2}L_8Sin[\frac{2\sqrt{e^{-A_2}f_2}}{A2^{\frac{3}{2}}}]))\Bigg];$$

$$L_3 = Csc[\frac{2(-\sqrt{f_2}+\sqrt{e^{-A_2}f_2})}{A_2^{\frac{3}{2}}}]Sin[\frac{2\sqrt{e^{-A_2}f_2}}{A_2^{\frac{3}{2}}}];$$

$$L_4 = Cos[\frac{2\sqrt{e^{-A_2}f_2}}{A_2^{\frac{3}{2}}}]Csc[\frac{2(-\sqrt{f_2}+\sqrt{e^{-A_2}f_2})}{A_2^{\frac{3}{2}}}];$$

$$L_5 = \frac{e^{\frac{2\sqrt{\frac{f_1}{A_1}}}{A_2}}}{e^{\frac{4\sqrt{\frac{f_1}{A_1}}}{A_2}} - e^{\frac{4\sqrt{\frac{f_1e^{-A_2}}{A_1}}}{A_2}}};$$

$$L_6 = \frac{e^{\frac{2\sqrt{\frac{f_1}{A_1}}+4\sqrt{\frac{e^{-A_2}f_1}{A_1}}}{A_2}}}{-e^{\frac{4\sqrt{\frac{f_1}{A_1}}}{A_2}} + e^{\frac{4\sqrt{\frac{f_1e^{-A_2}}{A_1}}}{A_2}}};$$

$$L_7 = e^{\frac{4\sqrt{A_1\sigma^2}}{A_2}} - e^{\frac{4\sqrt{A_1e^{A_2}\sigma^2}}{A_2}};$$

$$L_8 = \sqrt{A_1e^{A_2}\sigma^2}$$