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Research Article

FACE BIMAGIC LABELING ON SOME GRAPHS

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ABSTRACT: The existence of face bimagic labeling of types (1,0,1), (1,1,0) and (0,1,1) for double duplication of all vertices by edges of a ladder graph is proved. Also if G is (1, 0, 1) face bimagic, except for three sided faces then double duplication of all vertices by edges of G is face bimagic.

Keywords: double duplication graphs, face bimagic labeling.

AMS Subject Classification: 05C78

1 Introduction

In 1967 Rosa[7] introduced the concept of graph labeling. A graph labeling is assigning integers to the vertices or edges or both subject to specific conditions. Let G(V,E,F) be a graph whose vertex set, edge set and face set are |V| = v, |E| = e and |F| = f. A labeling of type (x, y, z) of G assigns labels from the set {1, 2, 3, …, xv+ye+zf} to vertex set, edge set and face set of G in such a way that each vertex will receive label x, each edge will receive label y and each face will receive label f and every label is used not more than once. The values of x, y and z are restricted to {0, 1}. The labelings of type (1,0,1), (1, 1, 0) and (0, 1, 1) represents vertex and face labelings, vertex and edge labelings and edge and face labelings respectively. The weight of a face $w_t(f)$ under a labeling is the sum of labels of face together with labels of vertices and edges forming that face.

Definition 1.1.[5] The double duplication of a vertex by an edge of a graph is defined as, a duplication of a vertex v_k by an edge e={$v_k'v_k''$} in a graph G produces a graph G′ in which N(v_k')={v_k, v_k''} and N(v_k'')={v_k,v_k'}. Again duplication of vertices v_k, v_k' and v_k'' by edges e′={u_kw_k}, e″={ $u_k'w_k'$ } and e‴={ $u_k''w_k''$ } respectively in G′ produces a new graph G″ such that, N(u_k) = {w_k, v_k }, N(w_k) = {u_k, v_k }, N(u_k') = { w_k', v_k'}, N(w_k') = { u_k', v_k'}, N(u_k'') = { w_k'', v_k''}, N(w_k'') = { u_k'', v_k''}. The double duplication of all vertices by edges of a graph G is denoted by DD_{VV}(G).

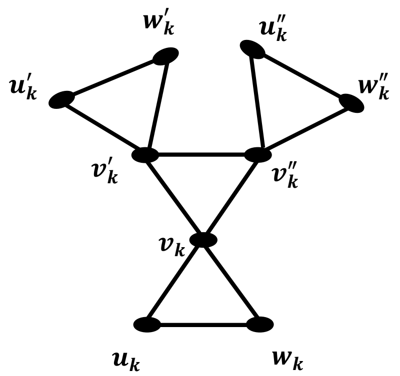

Figure 1. DD_{VV}(G)

Definition 1.2.[2] Let G = (V(G), E(G), F(G)) be a simple, finite, connected plane graph with the vertex set V(G), the edge set E(G) and the face set F(G). A bijection g from V(G) ∪E(G) ∪F(G) to the set {1, 2, …, |V(G)|+|E(G)|+|F(G)|} is called face bimagic if for every positive integer s the weight of every k-sided face is equal either to k_1 or to k_2.

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2 Main Results

Theorem 2.1

The graph DD_{VV} (L_n), n ≥3 of types (1,0,1), (1,1,0) and (0,1,1) is face bimagic.

Proof:

Let G be a ladder graph with vertex set V=$\{u_k, u'_k : 1 \le k \le n\}$, edge set E=$\{u_k u_{k+1}, u'_k u'_{k+1}: 1\le k \le n-1\} \cup \{u_k u'_k: 1 \le k \le n\}$ and face set F = $\{l_k : u_k u_{k+1} u'_k u'_{k+1} : 1\le k \le n-1\}$.

Let G′ be a graph obtained by a double duplication of a vertex by an edge in G with

$$V' = \{v_k, v'_k, w_k, w'_k, x_k, x'_k, y_k, y'_k, p_k, p'_k, q_k, q'_k, r_k, r'_k, s_k, s'_k: 1\le k \le n\} \cup V$$

$$E' = \{u_k v_k,\ u'_k v'_k,\ u_k w_k,\ u'_k w'_k,\ v_k w_k,\ v'_k w'_k,\ u_k r_k,\ u'_k r'_k,\ u_k s_k, u'_k s'_k, r_k s_k, r'_k s'_k,\ v_k x_k,\ v'_k x'_k,\ v_k y_k,\ v'_k y'_k,\ x_k y_k,\ x'_k y'_k,\ w_k p_k,\ w'_k p'_k,\ w_k q_k,\ w'_k q'_k,\ p_k q_k,\ p'_k q'_k : 1 \le k \le n\} \cup E \text{ and}$$

$$F' = \{f_k: u_k v_k w_k : 1\le k \le n\} \cup \{g_k : v_k x_k y_k : 1\le k \le n\} \cup \{h_k : w_k p_k q_k : 1\le k \le n\} \cup \{z_k: u_k r_k s_k : 1\le k \le n\} \cup \{f'_k: u'_k v'_k w'_k: 1\le k \le n\} \cup \{g'_k : v'_k x'_k y'_k: 1\le k \le n\} \cup \{h'_k: w'_k p'_k q'_k: 1\le k \le n\} \cup \{z'_k: u'_k r'_k s'_k: 1\le k \le n\} \cup F.$$

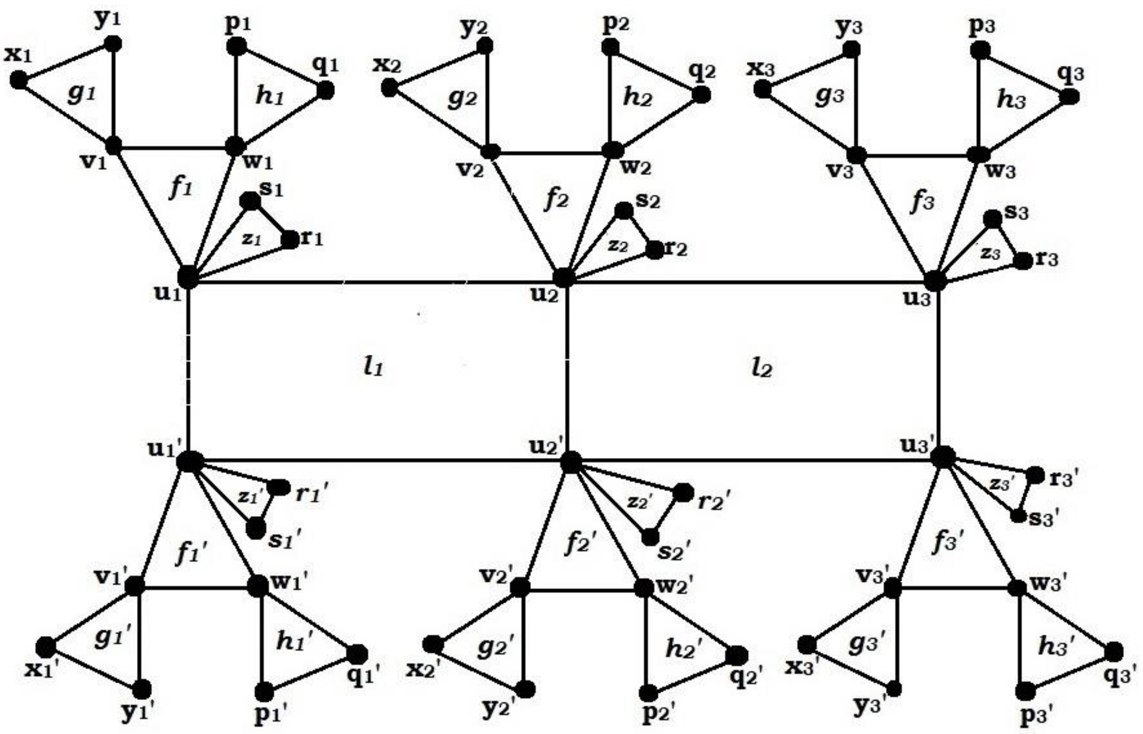

Figure 2. $DD_{VV}(L_3)$

The following are the face magic labeling of types (1,0,1), (1,1,0) and (0,1,1).

Type (i) : (1,0,1) - Face Magic

Define a mapping α_1 : V′∪F′ → {1,2,3,…,27n-1} as follows.

For $1 \le k \le n$,

$\alpha_1(u_k) = k$, $\alpha_1(v_k) = 4n+k$, $\alpha_1(w_k) = 4n+1-k$,
$\alpha_1(x_k) = 16n+k$, $\alpha_1(y_k) = 8n+1-k$, $\alpha_1(p_k) = 16n+1-k$,
$\alpha_1(q_k) = 8n+k$, $\alpha_1(r_k) = 12n+1-k$,, $\alpha_1(s_k) = 12n+k$,
$\alpha_1(f_k)= 26n+1-k$,, $\alpha_1(g_k) = 20n+1-k$, $\alpha_1(h_k) = 20n+k$,
$\alpha_1(z_k) = 24n+1-k$.

For $1 \le k \le n\text{-}2$, $\alpha_1(l_k) = 27n\text{-}1\text{-}k$, $\alpha_1(l_{n-1}) = 27n\text{-}1$.

Case(i): n ≡ 1(mod 2)

For $1 \le k \le \frac{n+1}{2}$

$\alpha_1(u'_{2k-1}) = 2n+1\text{-}k$, $\alpha_1(v'_{2k-1}) = 2n+k$,, $\alpha_1(w'_{2k-1}) = 6n+1\text{-}k$,
$\alpha_1(x'_{2k-1}) = 10n+1\text{-}k$, $\alpha_1(y'_{2k-1}) = 14n+k$, $\alpha_1(p'_{2k-1}) = 6n+k$,
$\alpha_1(q'_{2k-1}) = 18n+1\text{-}k$, $\alpha_1(r'_{2k-1}) = 14n+1\text{-}k$, $\alpha_1(s'_{2k-1}) = 10n+k$,
$\alpha_1(f'_{2k-1}) = 24n+k$, $\alpha_1(g'_{2k-1}) = 22n+1\text{-}k$, $\alpha_1(h'_{2k-1}) = 18n+k$,
$\alpha_1(z'_{2k-1}) = 22n+k$.

For $1 \le k \le \frac{n-1}{2}$

$\alpha_1(u'_{2k}) = \frac{3n+1}{2} - k$, $\alpha_1(v'_{2k}) = \frac{5n+1}{2} + k$, $\alpha_1(w'_{2k}) = \frac{11n+1}{2} - k$,
$\alpha_1(x'_{2k}) = \frac{19n+1}{2} - k$, $\alpha_1(y'_{2k}) = \frac{29n+1}{2} + k$, $\alpha_1(p'_{2k}) = \frac{13n+1}{2} + k$,
$\alpha_1(q'_{2k}) = \frac{35n+1}{2} - k$, $\alpha_1(r'_{2k}) = \frac{27n+1}{2} - k$, $\alpha_1(s'_{2k}) = \frac{21n+1}{2} + k$,
$\alpha_1(f'_{2k}) = \frac{49n+1}{2} + k$, $\alpha_1(g'_{2k}) = \frac{43n+1}{2} - k$, $\alpha_1(h'_{2k}) = \frac{37n+1}{2} + k$,
$\alpha_1(z'_{2k}) = \frac{45n+1}{2} + k$.

Case(ii): n ≡ 0(mod 2)

For $1 \le k \le \frac{n}{2}$

$\alpha_1(u'_{2k-1}) = \frac{3n}{2} +1\text{-}k$, $\alpha_1(v'_{2k-1}) = \frac{5n}{2} + k$, $\alpha_1(w'_{2k-1}) = \frac{11n}{2} +1\text{-}k$,,
$\alpha_1(x'_{2k-1}) = \frac{19n}{2} +1\text{-}k$, $\alpha_1(y'_{2k-1}) = \frac{29n}{2} + k$, $\alpha_1(p'_{2k-1}) = \frac{13n}{2} + k$,
$\alpha_1(q'_{2k-1}) = \frac{35n}{2} +1\text{-}k$, $\alpha_1(r'_{2k-1}) = \frac{27n}{2} +1\text{-}k$, $\alpha_1(s'_{2k-1}) = \frac{21n}{2} + k$,
$\alpha_1(f'_{2k-1}) = \frac{49n}{2} + k$, $\alpha_1(g'_{2k-1}) = \frac{43n}{2} +1\text{-}k$, $\alpha_1(h'_{2k-1}) = \frac{37n}{2} + k$,
$\alpha_1(z'_{2k-1}) = \frac{45n}{2} + k$, $\alpha_1(z'_{2k}) = 22n+k$.

$\alpha_1(u'_{2k}) = 2n+1\text{-}k$, $\alpha_1(v'_{2k}) = 2n+k$, $\alpha_1(w'_{2k}) = 6n+1\text{-}k$,
$\alpha_1(x'_{2k}) = 10n+1\text{-}k$, $\alpha_1(y'_{2k}) = 14n+k$, $\alpha_1(p'_{2k}) = 6n+k$,
$\alpha_1(q'_{2k}) = 18n+1\text{-}k$, $\alpha_1(r'_{2k}) = 14n+1\text{-}k$, $\alpha_1(s'_{2k}) = 10n+k$,
$\alpha_1(f'_{2k}) = 24n+k$, $\alpha_1(g'_{2k}) = 22n+1\text{-}k$, $\alpha_1(h'_{2k}) = 18n+k$.

Thus, the above labeling pattern gives the weight of all 3-sided and 4-sided faces as follows,
For $1 \le k \le n$,

The weight of all 3-sided faces is given by,

$w_1(g_k) = \alpha_1(v_k)+ \alpha_1(x_k)+ \alpha_1(y_k)+ \alpha_1(g_k) = 48n+2 = k_1$

Similarly, $w_1(g'_k) = w_1(h_k) = w_1(h'_k) = w_1(z_k) = w_1(z'_k) = 48n+2 = k_1$

$w_1(f_k) = \alpha_1(u_k)+ \alpha_1(v_k)+ \alpha_1(w_k)+ \alpha_1(f_k) = 34n+2 = k_2$

Similarly, $w_1(f'_k) = 34n+2 = k_2$

The weight of all 4-sided faces,
For $k = n\text{-}1$, $w_2(l_k) = \alpha_1(u_k) + \alpha_1(u_{k+1}) + \alpha_1(u'_k) + \alpha_1(u'_{k+1}) + \alpha_1(l_k) = k_1$

$$k_1 = \begin{cases} \frac{63n-1}{2}; & n \text{ is odd} \\ \frac{63n}{2}; & n \text{ is even} \end{cases}$$

For $1 \le k \le n\text{-}2$, $w_2(l_k) = \alpha_1(u_k) + \alpha_1(u_{k+1}) + \alpha_1(u'_k) + \alpha_1(u'_{k+1}) + \alpha_1(l_k) = k_2$

$$k_2 = \begin{cases} \frac{61n+1}{2}; & n \text{ is odd} \\ \frac{61n+2}{2}; & n \text{ is even} \end{cases}$$

Type(ii): (1,1,0) – Face Magic

Define a mapping $\alpha_2 : V' \cup E' \rightarrow \{1,2,3,\ldots, 45n-2\}$ as follows.

For $1 \leq k \leq n$,

$\alpha_2(u_k) = k$, $\alpha_2(v_k) = 4n+k$, $\alpha_2(w_k) = 4n+1-k$,
$\alpha_2(x_k) = 16n+k$, $\alpha_2(y_k) = 8n+1-k$, $\alpha_2(p_k) = 16n+1-k$,
$\alpha_2(q_k) = 8n+k$, $\alpha_2(r_k) = 12n+1-k$,, $\alpha_2(s_k) = 12n+k$,
$\alpha_2(u_kv_k) = 38n+k$, $\alpha_2(u_kw_k) = 42n+1-k$, $\alpha_2(v_kw_k) = 38n+1-k$,
$\alpha_2(v_kx_k) = 32n+1-k$, $\alpha_2(v_ky_k) = 28n+k$, $\alpha_2(x_ky_k) = 20n+1-k$,
$\alpha_2(w_kp_k) = 32n+k$, $\alpha_2(w_kq_k) = 28n+1-k$, $\alpha_2(p_kq_k) = 20n+k$,
$\alpha_2(u_kr_k) = 24n+k$, $\alpha_2(u_ks_k) = 36n+1-k$, $\alpha_2(r_ks_k) = 24n+1-k$.

For $1 \leq k \leq n-1$, $\alpha_2(u_ku_{k+1}) = 42n+k$, $\alpha_2(u'_ku'_{k+1}) = 44n-1-k$.
$\alpha_2(u_nu'_n) = 45n-2$.

Case(i): $n \equiv 1 \pmod 2$

For $1 \leq k \leq \frac{n+1}{2}$

$\alpha_2(u'_{2k-1}) = 2n+1-k$, $\alpha_2(v'_{2k-1}) = 2n+k$, $\alpha_2(w'_{2k-1}) = 6n+1-k$,
$\alpha_2(x'_{2k-1}) = 10n+1-k$, $\alpha_2(y'_{2k-1}) = 14n+k$, $\alpha_2(p'_{2k-1}) = 6n+k$,
$\alpha_2(q'_{2k-1}) = 18n+1-k$, $\alpha_2(r'_{2k-1}) = 14n+1-k$, $\alpha_2(s'_{2k-1}) = 10n+k$,

$\alpha_2(u'_{2k-1}v'_{2k-1}) = 40n+k$, $\alpha_2(u'_{2k-1}w'_{2k-1}) = 40n+1-k$, $\alpha_2(v'_{2k-1}w'_{2k-1}) = 36n+k$,
$\alpha_2(v'_{2k-1}x'_{2k-1}) = 26n+k$, $\alpha_2(v'_{2k-1}y'_{2k-1}) = 34n+1-k$,, $\alpha_2(x'_{2k-1}y'_{2k-1}) = 22n+1-k$,
$\alpha_2(w'_{2k-1}p'_{2k-1}) = 30n+1-k$, $\alpha_2(w'_{2k-1}q'_{2k-1}) = 30n+k$, $\alpha_2(p'_{2k-1}q'_{2k-1}) = 18n+k$,
$\alpha_2(u'_{2k-1}r'_{2k-1}) = 26n+1-k$, $\alpha_2(u'_{2k-1}s'_{2k-1}) = 34n+k$, $\alpha_2(r'_{2k-1}s'_{2k-1}) = 22n+k$.

For $1 \leq k \leq \frac{n-1}{2}$, $\alpha_2(u_{2k-1}u'_{2k-1}) = 45n-2-k$.

For $1 \leq k \leq \frac{n-1}{2}$

$\alpha_2(u'_{2k}) = \frac{3n+1}{2} - k$, $\alpha_2(v'_{2k}) = \frac{5n+1}{2} + k$, $\alpha_2(w'_{2k}) = \frac{11n+1}{2} - k$,
$\alpha_2(x'_{2k}) = \frac{19n+1}{2} - k$, $\alpha_2(y'_{2k}) = \frac{29n+1}{2} + k$, $\alpha_2(p'_{2k}) = \frac{13n+1}{2} + k$,
$\alpha_2(q'_{2k}) = \frac{35n+1}{2} - k$, $\alpha_2(r'_{2k}) = \frac{27n+1}{2} - k$, $\alpha_2(s'_{2k}) = \frac{21n+1}{2} + k$,

$\alpha_2(u'_{2k}v'_{2k}) = \frac{81n+1}{2} + k$, $\alpha_2(u'_{2k}w'_{2k}) = \frac{79n+1}{2} - k$, $\alpha_2(v'_{2k}w'_{2k}) = \frac{73n+1}{2} + k$,
$\alpha_2(v'_{2k}x'_{2k}) = \frac{53n+1}{2} + k$, $\alpha_2(v'_{2k}y'_{2k}) = \frac{67n+1}{2} - k$, $\alpha_2(x'_{2k}y'_{2k}) = \frac{43n+1}{2} - k$,
$\alpha_2(w'_{2k}p'_{2k}) = \frac{59n+1}{2} - k$, $\alpha_2(w'_{2k}q'_{2k}) = \frac{61n+1}{2} + k$, $\alpha_2(p'_{2k}q'_{2k}) = \frac{37n+1}{2} + k$,
$\alpha_2(u'_{2k}r'_{2k}) = \frac{51n+1}{2} - k$, $\alpha_2(r'_{2k}s'_{2k}) = \frac{45n+}{2} + k$, $\alpha_2(u'_{2k}s'_{2k}) = \frac{69n+1}{2} + k$.
$\alpha_2(u_{2k}u'_{2k}) = \frac{89n-3}{2} - k$.

Case(ii): $n \equiv 0 \pmod 2$

For $1 \leq k \leq \frac{n}{2}$

$\alpha_2(u'_{2k-1}) = \frac{3n}{2} + 1 - k$, $\alpha_2(v'_{2k-1}) = \frac{5n}{2} + k$, $\alpha_2(w'_{2k-1}) = \frac{11n}{2} + 1 - k$,,
$\alpha_2(x'_{2k-1}) = \frac{19n}{2} + 1 - k$, $\alpha_2(y'_{2k-1}) = \frac{29n}{2} + k$, $\alpha_2(p'_{2k-1}) = \frac{13n}{2} + k$,
$\alpha_2(q'_{2k-1}) = \frac{35n}{2} + 1 - k$, $\alpha_2(r'_{2k-1}) = \frac{27n}{2} + 1 - k$, $\alpha_2(s'_{2k-1}) = \frac{21n}{2} + k$,
$\alpha_2(u'_{2k-1}v'_{2k-1}) = \frac{81n}{2} + k$, $\alpha_2(u'_{2k-1}w'_{2k-1}) = \frac{79n+2}{2} - k$, $\alpha_2(v'_{2k-1}w'_{2k-1}) = \frac{73n}{2} + k$,
$\alpha_2(v'_{2k-1}x'_{2k-1}) = \frac{53n}{2} + k$, $\alpha_2(v'_{2k-1}y'_{2k-1}) = \frac{67n+2}{2} - k$, $\alpha_2(x'_{2k-1}y'_{2k-1}) = \frac{43n+2}{2} - k$,
$\alpha_2(w'_{2k-1}p'_{2k-1}) = \frac{59n+2}{2} - k$, $\alpha_2(w'_{2k-1}q'_{2k-1}) = \frac{61n}{2} + k$, $\alpha_2(p'_{2k-1}q'_{2k-1}) = \frac{37n}{2} + k$,

$\alpha_2(u'_{2k-1}r'_{2k-1}) = \frac{51n+2}{2} - k$, $\alpha_2(r'_{2k-1}s'_{2k-1}) = \frac{45n}{2} + k$, $\alpha_2(u'_{2k-1}s'_{2k-1}) = \frac{69n}{2} + k$.
$\alpha_2(u_{2k-1}u'_{2k-1}) = \frac{89n-2}{2} - k$.

$\alpha_1(u'_{2k}) = 2n+1-k$, $\alpha_1(v'_{2k}) = 2n+k$, $\alpha_1(w'_{2k}) = 6n+1-k$,
$\alpha_1(x'_{2k}) = 10n+1-k$, $\alpha_1(y'_{2k}) = 14n+k$, $\alpha_1(p'_{2k}) = 6n+k$,
$\alpha_1(q'_{2k}) = 18n+1-k$, $\alpha_1(r'_{2k}) = 14n+1-k$, $\alpha_1(s'_{2k}) = 10n+k$,
$\alpha_2(u'_{2k}v'_{2k}) = 40n+k$, $\alpha_2(u'_{2k}w'_{2k}) = 40n+1-k$, $\alpha_2(v'_{2k}w'_{2k}) = 36n+k$,
$\alpha_2(v'_{2k}x'_{2k}) = 26n+k$, $\alpha_2(v'_{2k}y'_{2k}) = 34n+1-k$, $\alpha_2(x'_{2k}y'_{2k}) = 22n+1-k$,
$\alpha_2(w'_{2k}p'_{2k}) = 30n+1-k$, $\alpha_2(w'_{2k}q'_{2k}) = 30n+k$, $\alpha_2(p'_{2k}q'_{2k}) = 18n+k$,
$\alpha_2(u'_{2k}r'_{2k}) = 26n+1-k$, $\alpha_2(r'_{2k}s'_{2k}) = 22n+k$, $\alpha_2(u'_{2k}s'_{2k}) = 34n+k$.

Thus, the above labeling pattern gives the weight of all 3-sided and 4-sided faces as follows,
For $1 \le k \le n$,
The weight of all 3-sided faces is given by,

$$w_1(g_k) = \alpha_2(v_k) + \alpha_2(x_k) + \alpha_2(y_k) + \alpha_2(v_kx_k) + \alpha_2(x_ky_k) + \alpha_2(v_ky_k) = 108n+3 = k_1$$

Similarly, $w_1(g'_k) = w_1(h_k) = w_1(h'_k) = w_1(z_k) = w_1(z'_k) = 108n+3 = k_1$

$$w_1(f_k) = \alpha_2(u_k) + \alpha_2(v_k) + \alpha_2(w_k) + \alpha_2(u_kv_k) + \alpha_2(u_kw_k) + \alpha_2(v_kw_k) = 126n+3 = k_2$$

Similarly, $w_1(f'_k) = 126n+3 = k_2$

The weight of all 4-sided faces,

For $k = n-1$, $w_2(l_k) = \alpha_2(u_k) + \alpha_2(u_{k+1}) + \alpha_2(u'_k) + \alpha_2(u'_{k+1}) + \alpha_2(u_ku_{k+1}) + \alpha_2(u'_ku'_{k+1}) + \alpha_2(u_ku'_k) + \alpha_2(u_{k+1}u'_{k+1}) = k_1$

$$k_1 = \begin{cases} \frac{359n-7}{2}; & n \text{ is odd} \\ \frac{359n-6}{2}; & n \text{ is even} \end{cases}$$

For $1 \le k \le n-2$, $w_2(l_k) = \alpha_2(u_k) + \alpha_2(u_{k+1}) + \alpha_2(u'_k) + \alpha_2(u'_{k+1}) + \alpha_2(u_ku_{k+1}) + \alpha_2(u'_ku'_{k+1}) + \alpha_2(u_ku'_k) + \alpha_2(u_{k+1}u'_{k+1}) = k_2$

$$k_2 = \begin{cases} 178n-4; & n \text{ is odd} \\ 179n-3; & n \text{ is even} \end{cases}$$

Type(iii): (0,1,1)

Define a function $\alpha_3 : E' \cup F' \to \{1,2,3,\ldots, 36n-3\}$ as follows,

For $1 \le k \le n-1$, $\alpha_3(u_ku_{k+1}) = k$, $\alpha_3(u'_ku'_{k+1}) = 2n-1-k$
For $1 \le k \le n$,

$\alpha_3(u_kv_k) = 7n-2+k$, $\alpha_3(u_kw_k) = 7n-1-k$, $\alpha_3(v_kw_k) = 3n-2+k$,
$\alpha_3(v_kx_k) = 23n-2+k$, $\alpha_3(v_ky_k) = 19n-1-k$, $\alpha_3(x_ky_k) = 11n-2+k$,
$\alpha_3(w_kp_k) = 23n-1-k$, $\alpha_3(w_kq_k) = 19n-2+k$ $\alpha_3(p_kq_k) = 11n-1-k$,
$\alpha_3(u_kr_k) = 27n-1-k$, $\alpha_3(u_ks_k) = 15n-2+k$, $\alpha_3(r_ks_k) = 15n-1-k$.

$\alpha_3(u'_kv'_k) = 5n-2+k$, $\alpha_3(u'_kw'_k) = 9n-1-k$, $\alpha_3(v'_kw'_k) = 5n-1-k$,
$\alpha_3(v'_kx'_k) = 21n-1-k$, $\alpha_3(v'_ky'_k) = 21n-2+k$, $\alpha_3(x'_ky'_k) = 9n-2+k$,
$\alpha_3(w'_kp'_k) = 17n-2+k$, $\alpha_3(w'_kq'_k) = 25n-1-k$, $\alpha_3(p'_kq'_k) = 13n-1-k$,
$\alpha_3(u'_kr'_k) = 17n-1-k$, $\alpha_3(u'_ks'_k) = 25n-2+k$, $\alpha_3(r'_ks'_k) = 13n-2+k$,

$\alpha_3(f_k) = 35n-1-k$, $\alpha_3(f'_k) = 33n-2+k$, $\alpha_3(g_k) = 31n-1-k$,
$\alpha_3(g'_k) = 33n-1-k$, $\alpha_3(h_k) = 31n-2+k$, $\alpha_3(h'_k) = 29n-2+k$,
$\alpha_3(z_k) = 27n-2+k$, $\alpha_3(z'_k) = 29n-1-k$, $\alpha_3(l_{n-1}) = 36n-3$.

For $1 \le k \le n-2$, $\alpha_3(l_k) = 36n-3-k$.

Case(i): $n \equiv 1 \pmod 2$

For $1 \le k \le \frac{n+1}{2}$, $\alpha_3(u_{2k-1}u'_{2k-1}) = 2n + \frac{n-5}{2} + k$, For $1 \le k \le \frac{n-1}{2}$, $\alpha_3(u_{2k}u'_{2k}) = 2n-2+k$,

Case(ii): n ≡ 0(mod 2)

For $1 \le k \le \frac{n}{2}$, $\alpha_3(u_{2k-1}u'_{2k-1}) = 2n+\frac{n-4}{2}+k$, $\alpha_3(u_{2k}u'_{2k}) = 2n-2+k$,

The following are the weights of all 3-sided and 4-sided faces of a ladder graph,
For $1 \le k \le n$,
The weight of all 3-sided faces is given by,
$w_1(g_k) = \alpha_3(v_kx_k)+\alpha_3(x_ky_k)+\alpha_3(v_ky_k)+\alpha_3(g_k) = 84n-6 = k_1$
Similarly, $w_1(g'_k) = w_1(h_k) = w_1(h'_k) = w_1(z_k) = w_1(z'_k) = 84n-6 = k_1$

Also, $w_1(f_k) = \alpha_3(u_kv_k)+\alpha_3(u_kw_k)+\alpha_3(v_kw_k)+\alpha_3(f_k) = 52n-6 = k_2$
Similarly, $w_1(f'_k) = 52n-6 = k_2$

The weight of all 4-sided faces,
For $k = n-1$,
$w_2(l_k) = \alpha_3(u_ku_{k+1}) + \alpha_3(u'_ku'_{k+1}) + \alpha_3(u_ku'_k)+\alpha_3(u_{k+1}u'_{k+1})+\alpha_3(l_k) = k_1$

$$k_1 = \begin{cases} \dfrac{87n-17}{2}; & n \text{ is odd} \\ \dfrac{87n-16}{2}; & n \text{ is even} \end{cases}$$

For $1 \le k \le n-2$,
$w_2(l_k) = \alpha_3(u_ku_{k+1}) + \alpha_3(u'_ku'_{k+1}) + \alpha_3(u_ku'_k)+\alpha_3(u_{k+1}u'_{k+1})+\alpha_3(l_k) = k_2$

$$k_2 = \begin{cases} \dfrac{85n-15}{2}; & n \text{ is odd} \\ \dfrac{85n-14}{2}; & n \text{ is even} \end{cases}$$

Hence the graph $DD_{VV}(L_n)$, $n \ge 3$ of types (1,0,1), (1,1,0) and (0,1,1) is face bimagic.

Theorem 2.2. If G is (1, 0, 1) face bimagic except for 3-sided faces then $DD_{VV}(G)$ is (1, 0, 1) face bimagic.

Proof:
By assumption the graph $G(V, E, F)$ with p vertices, e edges and g faces is face bimagic. Then there exists a mapping $\lambda : V \cup F \to \{1, 2, 3, \ldots, p+g\}$ such that the weight of each face is either k_1 or k_2 constant. The vertex set and the face set of G are,
$V = \{v_k, 1 \le k \le p\}$ and $F = \{g_k = p+x, 1 \le x \le g\}$.

Let G'(V', E', F') denotes the double duplication of all vertices by edges of G with,

$$V' = V \cup \{u_k, w_k, v'_k, u'_k, w'_k, v''_k, u''_k, w''_k : 1 \le k \le p\}$$
$$E' = E \cup \{v_ku_k, u_kw_k, v_kw_k, v_kv'_k, v_kv''_k, v'_ku'_k, v'_kw'_k, v''_ku''_k, v''_kw''_k, u''_kw''_k : 1 \le k \le p\}$$
$$F' = F \cup \{f_k : v_kv'_kv''_k, 1 \le k \le p\} \cup \{f'_k : u_kv_kw_k, 1 \le k \le p\} \cup \{f''_k : v'_ku'_kw'_k, 1 \le k \le p\}$$
$$\cup \{f'''_k : v''_ku''_kw''_k, 1 \le k \le p\}.$$

Consider a mapping $\lambda' : V' \cup F' \to \{1, 2, 3, \ldots, 13p+g\}$

To prove G' is face bimagic it is enough to prove (1, 0, 1) face bimagic for newly added vertices and edges.

For $1 \le k \le p$,

$\lambda'(v_k) = \lambda(v_k)$ $\lambda'(u_k) = 6p+g+k$, $\lambda'(w_k) = 6p+g+1-k$,

$\lambda'(v'_k) = 4p+g+1-k$, $\lambda'(u'_k) = 8p+g+1-k$, $\lambda'(w'_k) = 2p+g+k$,

$\lambda'(v''_k) = 8p+g+1-k$, $\lambda'(u''_k) = 4p+g+k$, $\lambda'(w''_k) = 2p+g+1-k$,

$\lambda'(f_k) = 12p+g\ k$, $\lambda'(f'_k) = 12p+g+1-k$, $\lambda'(f''_k) = 10p+g+1-k$,

$\lambda'(f_k''') = 10p+g+k.$

$\lambda'(g_k) = \lambda(g_k),\ 1 \le k \le g.$

The following are the weights for newly added faces:

$$\text{wt}\,[\lambda'(f_k)] = \lambda'(v_k) + \lambda'(v_k') + \lambda'(v_k'') + \lambda'(f_k)$$
$$= k+4p+g+1-k+8p+g+1-k+12p+g+k = 24p+3g+2$$

$$\text{wt}\,[\lambda'(f_k')] = \lambda'(v_k) + \lambda'(u_k) + \lambda'(w_k) + \lambda'(f_k')$$
$$= k+6p+g+k+6p+g+1-k+12p+g+1-k = 24p+3g+2$$

$$\text{wt}[\lambda'(f_k'')] = \lambda'(v_k') + \lambda'(u_k') + \lambda'(w_k') + \lambda'(f_k'')$$
$$= 4p+g+1-k+8p+g+k+2p+g+k+10p+g+1-k = 24p+4g+2$$

$$\text{wt}[\lambda'(f_k''')] = \lambda'(v_k'') + \lambda'(u_k'') + \lambda'(w_k'') + \lambda'(f_k''')$$
$$= 8p+g+1-k+4p+g+k+2p+g+1-k+10p+g+k = 24p+4g+2$$

Hence the resultant graph is (1, 0, 1) face bimagic for all 3-sided faces with $k_1 = 24p+3g+2$ and $k_2 = 24p+4g+2$.

3. Conclusion

In this paper, the face bimagic labeling of double duplication of all vertices by edges of a ladder graph along with a general result is studied. In future, this labeling technique can be used for real time applications like communication, radar fields etc.

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