Approximation of conjugate function related to Lipschitz and weighted class $(L_r, \xi(t))$ by product summability

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Abstract. **:**In the paper, a new theorem on the degree of approximation of the conjugate function corresponding to the Lip α and weighted $(L_r, \xi(t))$ class by product summabilitymeans of the conjugated Fourier series is proved.

Keywords: Approximation, Conjugate Fourier series, infinite series.

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1. Introduction

The degree of approximation of the functions in Lipschitz spaces using single and product summability means has been studied by the authors. But so far nothing seems to happen in the direction of the present work. So, we established a theorem on degree of approximation of the conjugate function corresponding to the Lip α and weighted $(L_r, \xi(t))$ class through the product sum of the conjugated Fourier series.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with a sequence of its n th partial sum $\{s_n\}$. .The change is defined as the nth partial sum of (C.I) the sum and is given by

$$c_n = \frac{1}{n+1}\sum_{k=0}^{n} s_n \to s \text{ as } n \to \infty.$$

So this $\sum_{n(=0)}^{\infty} u_n$ is is able to add a certain number of s by $(C,1)$ method.

If,

$$(R,q) = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} s_k \to s \text{ as } n \to \infty.$$

An infinite series $\sum_{n=0}^{\infty} u_n$ with partial sum s_n is then said to be summable by (R,q) the method for a fixed number s.A product of transform $(C,1)$ of (R,q) defines $(C,1)(R,q)$ transform and denotes by $C_n^1 R_n^q$ transform.

Thus if,

$C_n^1 R_n^q = (C,1)(R,q) = \frac{1}{n+1}\sum_{k=0}^{n} R_k^q \to s$ as $n \to \infty$. where R_n^q denotes the (R,q) transform, then the series $\sum_{n=0}^{\infty} u_n$ is called summable b $(C,1)(R,q)$ means or summable $(C,1)(R,q)$ to a definite number s. Let $g(x)$ be periodic and integral with period 2π in the sense of Lebesgue. The Fourier series is given by

$$g(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad (1.1)$$

The conjugate series of (1.1) is define as

$$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x) \qquad (1.2)$$

A signal (function) $g \in Lip\,\alpha$ is define as ,

$$g(x+t) - g(x) = O\left(|t|^{\alpha}\right) \text{ for } 0 < \alpha \le 1,$$

$$\left(\int_0^{2\pi} |g(x+t) - g(x)|^r dx\right)^{\frac{1}{r}} = O\left(|t|^{\alpha}\right), 0<\alpha \le 1, r \ge 1$$

And its positive increasing function $\xi(t)$,

$$\left(\int_0^{2\pi} |g(x+t) - g(x)|^r dx\right)^{\frac{1}{r}} = O(\xi(t)),$$

and when $g \in W(L_r, \xi(t))$ then,

$$\left(\int_0^{2\pi} \left| \{g(x+t) - g(x)\} \sin^{\beta} x\right| dx\right)^{\frac{1}{r}} = O(\xi(t)), \ \beta \ge 0.$$

When $\beta = 0$ then $W(L_r, \xi(t))$ coincides with $Lip(\xi(t), r)$ and when $\xi(t) = t^{\alpha}$ then $Lip(\xi(t), r)$ class coincides with the class $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ class reduce to the class $Lip\ \alpha$.

Norm defined by

$$\|g\|_r = \left(\int_0^{2\pi} |g(x)|^r dx\right)^{\frac{1}{r}}, \ r \ge 1 \qquad (1.3)$$

Degree of approximation $E_n(g)$ is define by

$$E_n = \min \|g - T_n\|_r \qquad (1.4)$$

Where $T_n(x)$ is the trigonometric polynomial of power degree n.

The following notation is used:

$$\phi(t) = g(x+t) - g(x-t) - 2g(x)$$

$$K_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left| \frac{1}{(1+q)^k} \sum_{k=0}^{n} \left\{ \binom{k}{\nu} q^{k-\nu} . \frac{\sin\left(\nu + \frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\} \right|$$

2. Main theorem

Lebesgue integral on $[0, 2\pi]$ and belong to $W(L_r, \xi(t))$ class, so its degree of approximation is given by

$$\left\| C_n^1 R_n^q - g \right\|_r = O\left[(n+1)^{\beta + \frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right] \qquad (2.1)$$

Provided $\xi(t)$ that the following condition is satisfies:

$\left\{\frac{\xi(t)}{t}\right\}$ be a decreasing order (2.2)

$$\left\{ \int_0^{\frac{\pi}{n+1}} \left(\frac{t|\phi(t)|}{\xi(t)} \right)^r \sin^{\beta r} t \, dt \right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \qquad (2.3)$$

and

$$\left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)} \right)^r \sin^{\beta r} t \, dt \right\}^{\frac{1}{r}} = O\left\{(n+1)^{\delta}\right\} \qquad (2.4)$$

Where δ is an arbitrary number such that $s(1-\delta) - 1 > 0$, $\frac{1}{r} + \frac{1}{s} = 1,$ the condition (2.3) and (2.4) holds equally in x and $C_n^1 R_n^q$ is $(C,1)(R,q)$ the means of the conjugated Fourier series (1.2).

2. Lemmas:

The following lemmas are necessary for our proof of main theorem:

Lemaa (3.1): $|K_n(t)| = O(n+1)$ for $0 \le t \le \frac{1}{n+1}$

Proof: For $0 \le t \le \frac{\pi}{n+1}$ $, \sin nt \le n \sin t$

$$K_n(t) \le \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left| \frac{1}{(1+q)^k} \sum_{k=0}^{n} \left\{ \binom{k}{\nu} q^{k-\nu} . \frac{(2\nu+1)\sin\frac{1}{2}t}{\sin\frac{t}{2}} \right\} \right|$$

$$\le \frac{1}{2\pi(n+1)} \left| \sum_{k=0}^{n} \left[\frac{1}{(1+q)^k} (2k+1) \sum_{\nu=0}^{k} \binom{k}{\nu} q^{k-\nu} \right] \right|$$

$$= \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} (2k+1)$$

$$= O(n+1).$$

Lemma (3.2): $|K_n(t)| = O\left(\frac{1}{t}\right)$ for $\frac{\pi}{n+1} \le t \le \pi.$

Proof: For $\frac{\pi}{n+1} \le t \le \pi.$, using Jordan's lemma, $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$ and $\sin(nt) \le 1.$

$$K_n(t) \le \frac{1}{2\pi(n+1)} \left| \sum_{k=0}^{n} \left[\frac{1}{(1+q)^k} (2k+1) \sum_{\nu=0}^{k} \binom{k}{\nu} q^{k-\nu} . \frac{1}{\left(\frac{t}{\pi}\right)} \right] \right|$$

$$\le \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left[\frac{1}{(1+q)^k} (2k+1) \right]$$

$$= \frac{1}{2t(n+1)} \sum_{k=0}^{n} 1$$

4. Proof

Following Titchmarsh [20] and using the Reimann – Lebesgue theorem, $s_n\{g;x\}$ of the series (1.2) is given

$$s_n\{g;x\} - f(x) = \frac{1}{2\pi} \int_0^{\pi} \phi(t) \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \, dt$$

Therefore, conversion of (R, q) transform $\left(R_n^q\right)$ is $s_n\{g;x\}$ using (1.2) is given by

$$R_n^q - g(x) = \frac{1}{2\pi(1+q)^k} \int_0^{\pi} \frac{\phi(t)}{\sin\frac{t}{2}} \left\{ \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin\left(k+\frac{1}{2}\right)t \right\} dt \ .$$

Denoting $(C,1)(R,q)$ transform of $s_n\{g;x\}$ as $C_n^1 R_n^q$ the change as now, we write

$$C_n^1 R_n^q - g(x) = \frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\left\{\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sin\left(\nu+\frac{1}{2}\right)t\right\}dt\right]$$

$$=\left[\int_0^{\frac{\pi}{n+1}}+\int_{\frac{\pi}{n+1}}^{\pi}\right]\phi(t)K_n(t)\,dt \ = I_1+I_2, \qquad say \qquad (4.1)$$

Now,

$$|I_1| \le \int_0^{\frac{\pi}{n+1}}|\phi(t)||K_n(t)|\,dt$$

We have,

$$|\phi(x+t)-\phi(x)| \le |g(u+x+t)-g(u+x)| + |g(u-x-t)-g(u-x)|$$

Using Minkowski'sinequality ,

$$\left[\left|\int_0^{2\pi}\{\phi(x+t)-\phi(x)\}\sin^{\beta}x\right|^r dx\right]^{\frac{1}{r}} \le \left[\int_0^{2\pi}\left|\{g(u+x+t)-g(u+x)\}\sin^{\beta}x\right|^r dx\right]+$$

$$\left[\int_0^{2\pi}\left|\{g(u+x-t)-g(u-x)\}\sin^{\beta}x\right|^r dx\right]$$

$$=O(\xi(t))$$

$$=O\left(\frac{1}{t}\right).$$

Then $g\in W(L_r,\xi(t)) \Rightarrow \phi(t)\in W(L_r,\xi(t))$.

Using Holder's inequality and the fact that $\phi(t)\in W(L_r,\xi(t))$, ,(2.3), the lemma (3.1) and the second mean value theorem for integrals,

Therefore ,

$$|I_1| \le \left[\int_0^{\frac{\pi}{n+1}}\left\{\frac{t|\phi(t)|\sin^{\beta}t}{\xi(t)}\right\}^r dt\right]\left[\int_0^{\frac{\pi}{n+1}}\left\{\frac{\xi(t)|K_n(t)|}{t\sin^{\beta}t}\right\}^s dt\right]^{\frac{1}{s}}$$

$$=O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{\pi}{n+1}}\left\{\frac{(n+1)\xi(t)}{t^{1+\beta}}\right\}^s dt\right]^{\frac{1}{s}}$$

$$=O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\varepsilon}^{\frac{\pi}{n+1}}\left\{\frac{(n+1)\xi(t)}{t^{(1+\beta)s}}\right\}^{s}dt\right]^{\frac{1}{s}} \quad for\ \ some\ \ 0<\varepsilon<\frac{1}{n+1}$$

$$=O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\},\ \sin ce\ \frac{1}{r}+\frac{1}{s}=1 \qquad (4.2)$$

Now Hölder's , $\left|\sin t\right|<1,\ \sin t\geq\left(\frac{2t}{\pi}\right)$, lemma (3.2),(2.2),(2.4) and mean value theorem.

$$\left|I_2\right|\leq\left[\int_{\frac{\pi}{n+1}}^{\pi}\left\{\frac{t^{-\delta}\ \left|\phi(t)\right|\sin^{\beta}t}{\xi(t)}\right\}^{r}\right]dt\ \left[\int_{\frac{\pi}{n+1}}^{\pi}\left\{\frac{\xi(t)\left|K_n(t)\right|}{t^{-\delta}\ \sin^{\beta}t}\right\}^{s}dt\right]^{\frac{1}{s}}$$

$$=O\left\{(n+1)^{\delta}\right\}\left[\int_{\frac{1}{\pi}}^{n+1}\left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-1-\beta}}\right\}^{s}\frac{dy}{y^{2}}\right]^{\frac{1}{s}}$$

$$=O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\frac{1}{\pi}}^{n+1}\frac{dy}{y^{s(\delta-1-\beta)+2}}\right]^{\frac{1}{s}}$$

$$=O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{s(\delta-1-\beta)+1}-\pi^{s(\delta-1-\beta)+1}\right]^{\frac{1}{s}}$$

$$=O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(1+\beta=\delta)-\frac{1}{s}}\right]^{\frac{1}{s}}$$

$$=O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} \quad \because\ \frac{1}{r}+\frac{1}{s}=1 \qquad (4.3)$$

Now collecting (4.1), (4.2) and (4.3) we get

$$\left|C_n^1\,R_n^q-g(x)\right|=O\left\{(n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right\}$$

$$\left\|C_n^1\,R_n^q-g(x)\right\|_r=O\left\{\int_0^{2\pi}O\left\{(n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right\}^{r}dx\right\}^{\frac{1}{r}}$$

$$=O\left\{(n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\int_0^{2\pi}dx\right\}^{\frac{1}{r}}$$

$$=O\left\{(n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right\}$$

Which is the complete proof of the theorem.

5. Corollary

Following corollary becomes particular case of our main theorem.

5.1. Corollary

If $\beta=0$ and $\xi(t)=t^{\alpha}$, then the degree of approximation of a function is given by

$$\left\|C_n^1 R_n^q - g(x)\right\|_r = O\left\{\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right\}, \quad g\in Lip(\alpha,r), 0<\alpha\le 1,.$$

5.2. Corollary

If $r\to\infty$ in above corollary (5.1), then for $0<\alpha<1$is given by

$$\left\|C_n^1 R_n^q - g(x)\right\|_r = O\left\{\frac{1}{(n+1)^{\alpha}}\right\}$$

5.3. *Corollary*

If $\beta=0$, $\xi(t)=t^{\alpha}$ and $q_n=1, \forall n$, then the degree of approximation of a function $g\in Lip(\alpha,r), 0<\alpha\le 1$, is given by

$$\left\|C_n^1 R_n^q - g(x)\right\|_r = O\left\{\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right\}$$

5.4 .Corollary

If $r\to\infty$ in above corollary (5.3), then for $0<\alpha<1$, we have

$$\left\|C_n^1 R_n^q - g(x)\right\|_{\infty} = O\left\{\frac{1}{(n+1)^{\alpha}}\right\}.$$

6. Applications

These approximations have wide applications in signal analysis [12] and digital signal processing [13]. Engineers and scientists use the properties of the Fourier approximation to design digital filters. In particular and Moustakides [13] presented a new L_2 based method for designing finite impulse response digital filters and obtained corresponding optimal estimates with improved performance.

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