The Spectral Petrov-Galerkin Method for Solving Integral Equations of the First Kind

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Abstract: The major goal of this research is to use the spectral Petrov-Galerkin method (SPGM) to solve the first type of Volterra integral equations (VIEs). Three examples will be given to demonstrate how the suggested technique works, in comparison to the examples in [1] and [11], where the numerical and exact solutions are quite harmonic, effectively indicating the numerical answers. The absolute errors were determined to estimate the validity of the suggested method and to demonstrate that it was an effective and valid method.

Keywords: integral equation, spectral method, Petrov-Galerkin method, Laguerre polynomial, convergence.

1. Introduction

Physical issues from all disciplines are frequently turned into integral equations, especially Volterra linear integral equations of the first kinds, respectively. Several writers have investigated and applied these equations in order to arrive at an analytic and unique numerical solution. In recent years, there has been an increase in interest in VIEs, owing to their wide range of applications in mathematical physics (astrophysics, contact problem, heat transfer problem, and reactor theory). As a result, since the introduction of the digital computer a few decades ago, most traditional analytic integral equations solvers have been developed and constructed. Many researchers have created numerical methods for the solution of VIEs using various polynomials, and more researchers have developed numerical methods for the solution of VIEs using various polynomials. Islam [14] employed Hermite and Chebyshev polynomials to solve linear and nonlinear VIEs. For the numerical solution of VIEs of the second kind. This work is concerned with the following first kind of VIEs [7].

$$\int_a^x k(x,t)u(t)dt = f(x)\,, \qquad x \in [a,b]. \qquad (1.1)$$

where the kernel function $k(x,t)$ and the source function $f(x)$are given smooth functions, $u(t)$ is the unknown function.

In fact, a simple linear employed in [14] may be used to write any first kind of VIE with smooth kernel into (1.1). As a result, our method can be used to VIEs. In any interval where the kernel is smooth, this is defined. We'll look at the problem where the solution is (1.1). As a result, it's only appropriate to use very high-order numerical approaches like spectral methods to solve problems (1.1). It is well known that there exist numerical techniques for solving (1.1), such as collocation methods and the finite element method [3], but only a few papers have addressed spectral approximations.The Chebyshev spectral methods are investigated in [12] for the VIEs under multiple-precision arithmetic, in [10] the application of the Chebyshev polynomial for solving Fredholm integral equations. Some efforts are made to implement the spectral methods to solve the VIEs.Actually, the success of the spectral method for VIEs is the main motivation for our work in the first kind of integral equations.

2 Orthogonal polynomials

Mathematicians have been interested in orthogonal polynomials in several areas. This interest has often come from outside the polynomials community in recent years, with the discovery of their relationship to inetegrable systems [9]

Let $\int_a^b w(x)\varphi_i(x)\varphi_j(x) = \delta_{ij}$ (2.1)

With the Kronecker deltaδ_{ij} defined by

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

where the weight function $w(x)$ is continuous and positive on [a, b] such that the moments exist.

Then the inner product of polynomials φ_i and φ_j given by:

$\langle \varphi_i, \varphi_j \rangle = \int_a^b w(x)\varphi_i(x)\varphi_j(x)dx$ (2.2)

for orthogonality

$\langle \varphi_i, \varphi_j \rangle = 0, i \neq j$ (2.3)

In this study, we adopt the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ in the interval $[a, b]$.

The construction of φ_i, $i = 0,1,2,\cdots$ of approximant:

$u_N(x) = \sum_{j=0}^{N-1} u_j(\varphi_j(x) + s_j\varphi_{j+1}(x)) \cong u(x)$ (2.4)

where s_j constant and φ_j basis function $j = 0,1,2,\cdots,N-1$.

In this work a trail function of basis Laguerre polynomial and a test function of basis Chebyshev polynomial with weight function in this work.

2.1 Laguerre polynomials

2.1.1 Defintion

The Laguerre polynomials $L_m(x)$ are solutions of the Laguerre differential equations and consist of a set of orthogonal polynomials all over the interval [0, ∞] expresses the explicit formula for $L_m(x)$ [3,2,6] .

$L_m(x) = \sum_{i=0}^{m} (-1)^i \frac{m!}{(m-i)!(i!)^2} x^i$, (2.5)

The first four of Laguerre polynomials are

$L_0(x) = 1$, $L_1(x) = x$, $L_2(x) = \frac{1}{2!}(x^2 - 4x + 2)$, $L_3(x) = \frac{1}{3!}(-x^3 + 9x^2 - 18x + 6)$,

$L_4(x) = \frac{1}{4!}(x^4 - 16x^3 + 72x^2 - 96x + 24)$,

So, the recurrence relationship is

$(m+1)L_{m+1}(x) = (2m + 1 - x)L_m(x) - mL_{m-1}(x)$ (2.6)

Rodrigues formula also can be defined as :

$L_m(x) = \frac{e^x}{m!}\frac{d^m}{dx^m}(e^{-x}x^m)$ (2.7)

for the weigh function $w(x) = e^{-x}$ the Laguerre polynomials with the orthogonal property

2.2 Chebyshev polynomials

2.2.1 Defintion

The Chebyshev polynomials are a recursively defined sequence of orthogonal polynomials connected to de Moiver's formula. The Chebyshev polynomials of nth degree have a generic form specified by [15]

$T_n(x) = \sum_{m=0}^{n/2} (-1)^m \frac{n!}{(2m)!(n-2m)!}(1 - x^2)^m x^{n-2m}$ (2.8)

Where

$$\left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2} & \text{if n is even} \\ \frac{n+1}{2} & \text{if n is odd} \end{cases}$$

the first four Chebyshev polynomials of the first kind are :

$T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$

3 The implement of the Spectral Petrov-Galerkin method

By introducing the integral operator k defined as

$$ku(x) = \int_a^x k(x,t)u(t)dt,$$

Eq.(1.1) can be express as follows:

$ku(x) = g(x)$, $x \in [0,1]$ (3.1)

We will adopt the SPGM to solve this underlying problem.

First, let's look at how SPGM is implemented numerically. P_N is a space containing polynomials defined on [0,1] with a maximum degree of N, $\varphi_j(x)$ is the jthe Laguerre polynomial corresponding to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$, with $j = 0,1,\cdots,N$. As a consequence,

$$P_N = \text{span}\,\{\varphi_0(x), \varphi_1(x), \cdots, \varphi_N(x)\}.$$

Define the polynomial space V_N as show,

$$V_N = \{u: u\epsilon P_N\}.$$

Find $u_N \in V_N$ such that

$$(ku_N, v_N)_w = (g, v_N)_w, \qquad \forall v_N \in P_{N-1} \quad (3.2)$$

Where

$$(ku, v)_w = \int_0^1 \int_0^x k(x,t)u(t)v(x)w(x)dtdx$$

is continuous inner product, set

$$u_N(x) = \sum_{j=0}^{N-1} u_j\,(\varphi_j(x) + s_j\,\varphi_{j+1}(x)),$$

when $\xi_i(x)$, $i = 0,1,\cdots,N-1$ is a Chebyshev polynomial test function with a weight function from space, Laguerre polynomial $\varphi_j(x)$, $j = 0,1,\cdots,N-1$ is used. We get the following result from (2.2)

$$\sum_{j=0}^{N-1}(\xi_i(x), k(\varphi_j(x) + \varphi_{j+1}(x)))_w u_j = \left(\xi_i(x), g(x)\right)_w \qquad (3.3)$$

$$AU_{N-1} = g_{N-1}, \qquad (3.4)$$

where

$$a(i,j) = \int_0^1 \left[\int_0^x k(x,t)\left(\varphi_j(t) + \varphi_{j+1}(t)\right)dt\right]\xi_j(x)w(x)dx,\ g_{N-1}(i) = \int_0^1 \xi_i(x)g(x)w(x)dx$$

$$\begin{bmatrix} (\xi_0,\varphi_0+\varphi_1)_w & (\xi_0,\varphi_1+\varphi_2)_w & (\xi_0,\varphi_2+\varphi_3)_w & \cdots & (\xi_0,\varphi_{N-1}+\varphi_N)_w \\ (\xi_1,\varphi_0+\varphi_1)_w & (\xi_1,\varphi_1+\varphi_2)_w & (\xi_1,\varphi_2+\varphi_3)_w & \cdots & (\xi_1,\varphi_{N-1}+\varphi_N)_w \\ (\xi_2,\varphi_0+\varphi_1)_w & (\xi_2,\varphi_1+\varphi_2)_w & (\xi_2,\varphi_2+\varphi_3)_w & \cdots & (\xi_2,\varphi_{N-1}+\varphi_N)_w \\ (\xi_3,\varphi_0+\varphi_1)_w & (\xi_0,\varphi_1+\varphi_2)_w & (\xi_3,\varphi_2+\varphi_3)_w & \cdots & (\xi_3,\varphi_{N-1}+\varphi_N)_w \\ & & . & & \\ & & . & & \\ & & . & & \\ (\xi_{N-1},\varphi_0+\varphi_1)_w & (\xi_{N-1},\varphi_1+\varphi_2)_w & (\xi_{N-1},\varphi_2+\varphi_3)_w & \cdots & (\xi_{N-1},\varphi_{N-1}+\varphi_N)_w \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ . \\ . \\ . \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} (g,\xi_0) \\ (g,\xi_1) \\ (g,\xi_2) \\ (g,\xi_3) \\ . \\ . \\ . \\ (g,\xi_{N-1}) \end{bmatrix}$$

4 Theoretical analysis of SPGM to linear VIEs first kind

Define a weighted space as [4]

$L_w^2(I) = \{v: v \text{ is measureable and } \|v\|_w < \infty\}$,

where,

$$I \subseteq [a,b]$$

$\|v\|_w = (\int_a^b w(x)v^2(x)dx)^{\frac{1}{2}}$.

Further, define

$H_w^m(I) = \{v: D^k v \in L_w^2(I), \quad 0 \le k \le m\}$,

equipped with norm

$\|v\|_{m,w} = (\sum_{k=0}^m \|D^k v\|_w^2)^{1/2}$,

with $D^k v = \frac{d^k v}{dx^k}$.

When $w(x) = \frac{1}{\sqrt{1-x^2}}$, $L_w^2(I)$, $H_w^m(I)$ and $\|.\|_w$ are denoted simply by $L^2(I)$,$H^m(I)$ and $\|.\|$, respectively.

First, we define the orthogonal projection $\pi_N: L_w^2(I) \to P_N$ such that for any $u \in L_w^2(I)$.

Let B be a Banach space with the norm $\|.\|$ and B^* be its dual space of continuous linear functional .for each positive integer n, we assume that $A_n \subset B, B_n \subset B^*$, and A_n, B_n are finite dimensional vector spaces with dim $A_n = \dim B_n$ also A_n,B_n satisfy condition (H) : for each $x \in B$ and $y \in B^*$, there exist $x_n \in A_n$, $y_n \in B_n$ such that $\|x_n - x\| \to 0$ as $n \to \infty$ and $\|y_n - y\| \to 0$ as $n \to \infty$, when the SPGM for Eq.(1.1) is a numerical method for find $u_n \in B_n$ such that :

$(ku_N, v_N)_w = (g, v_N)_w, \qquad \forall\, v_n \in B_n \qquad (4.1)$

Definition 4.1:[9] For each x ∈ B, an element $P_n x \in A_n$ is called the generalized best approximation from A_n to x with respect to B_n, by the equation

$$\langle x - p_n x, v_n\rangle = 0\,, \qquad \forall\, v_n \in B_n$$

Definition 4.2: [8] $\{A_n, B_n\}$ are called a regular pair if a linear operator $P_n : A_n \to B_n$ exist and $P_n A_n = B_n$ also satisfying the following conditions:

(H_1): $\|x_m\| \le C_1 \langle x_n, P_n x_n\rangle^{1/2}$, $\quad \forall\, x_n \in A_n$

(H_2): $\|P_n x_m\| \le C_2 \|x_n\|$, $\quad \forall\, x_n \in A_n$

Where C_1 and C_2 are positive constant independent of n . If a pair of sequence $\{A_n\}$ and $\{B_n\}$ satisfies (H_1) and (H_2) , we call $\{A_n, B_n\}$ a regular pair. If a regular pair $\{A_n, B_n\}$ satisfies $\dim A_n = \dim B_n$ and condition (H), then the corresponding generalized projection P_n satisfies:

(1) For all $x \in B$, $\|P_n x - x\| \to 0$ as $n \to \infty$,

(2) There is constant $C > 0$ such that, $\|P_n\| < C$, $\qquad n = 1,2,\ldots$

(3) For some constant $C > 0$ independent of n, $\quad \|P_n x - x\| \le C\|u_n x - x\|$

where u_n is the best approximation from A_n to x.

The SPGM for equation (3.2) is a numerical method for finding $u_n \in A_n$ such that:

$(ku_n, v_n)_w = (f, v_n)_w\ \forall\, v_n \in B_n \quad (4.2)$

If $\{A_n, B_n\}$ is a regular pair with a linear operator $P_n : A_n \to B_n$, then equation (4.2) become as :

$(ku_n, P_n x_n) = (f, P_n x_n)\ \forall\, x_m \in A_m \quad (4.3)$

furthermore, equation (4.3)

$$P_n k u_n = P_n f$$

Now, assume $u_n \in A_n$ and $\{\varphi_j + \varphi_{j+1}\}_{j=0}^{N-1}$ are basis for A_n is trail space and $\{\xi_i\}_{i=0}^{N-1}$ are test space is a basis for B_n with weight function. Therefore the SPGM on [a , b] for equation (3.2) is :

$$(ku_n, \xi_i)_w = (f, \xi_i)_w \quad , \qquad i = 0,1,\cdots, N-1$$

Let $\qquad u_N(x) = \sum_{j=0}^{N-1} u_j\,(\varphi_j(x) + \varphi_{j+1}(x)),$

Then equation (3.2) leads to determining $\{u_i\}_{i=0}^{N-1}$ as the solution of the linear system:

$$\int_a^b \left(\int_a^x k(x,t) u_n(t)\,dt\right) \xi_i(x) w(x) dx = \int_a^b f(x)\xi_i(x)w(x)dx \quad i = 0,1,2,\cdots, N-1$$

$$\sum_{j=0}^{N-1} u_j \Big(\int_a^b \int_a^x k(x,t)\big(\varphi_j(t) + \varphi_{j+1}(t)\big)\, \xi_i(x) w(x) dt\Big) dx = \int_a^b f(x)\xi_i(x)w(x)dx$$

The SPGM using regular pair $\{A_n, B_n\}$ of piecewise polynomial space are called Petrov-Galerkin elements

$F_j = \int_a^b f(x)\xi_j(x)dx\,, \quad A_{ij} = \int_a^b \int_a^x k(x,t)(\varphi_j(t) + \varphi_{j+1}(t))\xi_i(x)w(x)dtdx$

Proposition 4.1:[5] For each $x \in X$, the generalized best approximation from X_n to x with respect to Y_n exists uniquely if and only if

$Y_n \cap X_n^{\perp} = \{0\} \quad (4.4)$

under this condition, P_n is a projection; i.e., $P_n^2 = P_n$.

Proof. Since dim X_n =dim Y_n. we assume that X_n and Y_n have bases $\{\varphi_i\}_{i=0}^{N}$ and $\{\xi_i\}_{i=0}^{N}$.

Let x ∈ X be given, to show that is a unique $P_n x \in X_n$ satisfying equation (3.3), we equivalently prove that linear system

$\sum_{i=0}^{N-1} c_i \langle \varphi_i, \xi_j\rangle = \langle x, \xi_j\rangle \qquad i, j = 0,1,2,\cdots, N-1 \qquad (4.5)$

Has a unique solution $\{c_i\}_{i=0}^{N-1}$. This is equivalent to showing that coefficient matrix $A = \langle \varphi_i, \xi_j\rangle$ is nonsingular.

To prove the exists $y_n \in Y_n \cap X_n^{\perp}$. Since $y_n \in Y_n \cap X_n^{\perp}$. Since $y_n \in Y_n$, we can write $y_n = \sum_{j=0}^{N-1} c_j \xi_j$ .By the fact that$y_n \in X_n^{\perp}$, we have

$$\sum_{j=0}^{N-1} c_j\langle \varphi_i, \xi_j\rangle = 0, \qquad i = 0,1,\cdots,N-1.$$

Since the matrix A is nonsingular, then $c_j = 0$ for $j = 0,1,2,\cdots,N-1$.Thus, $y_n = 0$ and $Y_n \cap X_n^{\perp} = \{0\}$.

Conversely, assume A is nonsingular, then $\{c_i\}_{i=0}^{N-1}$, not all zero, such that

$$\sum_{j=0}^{N-1} c_j\langle \varphi_i, \xi_j\rangle = 0,\ i = 0,1,\cdots,N-1.$$

Let $y_n = \sum_{j=0}^{N-1} c_j\,\xi_j$. Thus, $y_n \neq 0$ and $y_n \in Y_n \cap X_n^{\perp}$. This implies that $Y_n \cap X_n^{\perp} \neq \{0\}$.

Now to show that P_n is a projection, have just proved that under condition (3.4), every $x \in X$, we have$P_n x \in X_n$ that satisfies equation (3.3). For any $x \in X$, we have $P_n x \in X_n \subseteq X$, thus, by definition,

$$\langle P_n x - P_n^2 x, y_n\rangle = 0 \forall\ y_n \in Y_n.$$

From this equation and (3.3), we find that $P_n^2 x \in X_n$satisfies

$$\langle x - P_n^2 x, y_n\rangle = 0\ \forall y_n \in Y_n$$

By the uniqueness, we conclude that every $x \in X$

$$P_n^2 x = P_n x.$$

That is P_n is a projection.

Proposition 4.2:[5] Suppose that there is a linear operator $\pi_n X_n = Y_n$ and

$\|X_n\| \leq C_n\langle x_n, \pi_n x_n\rangle^{\frac{1}{2}} \forall x_n \in X_n,$

where the constant $C_n > 0$, depend on n but not on x_n. Then, equation (3.4) holds, thus, every $x \in X$ has a unique best approximation from X_n with respect to Y_n.

Proof. Let $y_n \in Y_n \cap X_n^{\perp}$. Since $\pi_n X_n = Y_n$. For this particular y_n there exists$x_n \in X_n$ such that

$\pi_n X_n = Y_n$. By assumption.

$$\|x_n\| \leq C_n\langle x_n, \pi_n X_n\rangle^{\frac{1}{2}} = C_n\langle x_n, y_n\rangle^{\frac{1}{2}} = 0.$$

The last equality holds because $y_n \in X_n^{\perp}$. This implies that $x_n = 0$. Thus $y_n = \pi_n X_n = 0$. Which show that equation (4.4) holds.

Theorem 4.1:[9]Let X be a Banach space and $k: X \to X$be a compact linear operator, assume that 1 is not an eigenvalue of the operatorK, suppose that X_n and Y_n satisfy condition (H) and $\{X_n, Y_n\}$ is regular pair Then there exists an $N_0 > 0$ such that for $n > N_0$, equation (3.6) has a unique solution $u_n \in X_n$ for any given $f \in X$that satisfies

$$\|u_n - u\| \leq C \inf_{x_n \in X_n} \|u - x_n\|, \qquad n > N_0$$

Where $u \in X$ is the unique solution (1.1) and $C > 0$ is a constant independent of n.

Proof. By Proposition 4.1p_n converges point wise to identity operator I in x. Hence, it follows from Theorem 4.1 that there exists an integer $N_0 > 0$ for which

$$\|u_n - u\| \leq C\|p_n u - u\|, \qquad n > N_0$$

Using this estimate and Proposition 4.1 we conclude the statement of this theorem. Assume that X is a Hilbert space and let $X_n = Y_n$. In this case, PGM. It has been pointed out earlier that let π_n be the identity operator in X, then $\{X_n, X_n\}$ is a regular pair. In other words, in the case of the PGM, the conditions for which $\{X_n, X_n\}$ is a regular pair are trivially satisfied, thus Theorem 3.6 holds for the PGM. Finally, we show that PGM in a Hilbert space X is also an example of the PGM, assume $K: X \to X$ is and denote $A := I - K$. Let X_n= $\mathrm{span}\{\varphi_1, \ldots, \varphi_N\} \subseteq X$with $\dim X_n = N$.In the PGM.

$\langle Ku_n, y_n\rangle = \langle f, y_n\rangle$ for all $y_n \in Y_n$.

We let $Y_n :=$span $\{A\varphi_1, \cdots, A\varphi_N\}$. Then we obtain

$\langle Au_n - f, A\varphi_l\rangle = 0, l = 1, \cdots, N-1$ (4.6)

This linear system gives the PGM (1.1), which is to find $u_n \in X_n$ such that

$$\|Au_n - f\| = \inf_{x_n \in X_n} \|Ax_n - f\|.$$

On the other hand, equation (3.6) is equivalent to the following:

$$\langle A^*Au_n - A^*f, \varphi_l\rangle = 0$$

That is, the equation (1.1) is equivalent to the PGM for the equation

$$I - \widehat{K}u = A^*f,$$

where

$$\widehat{K} := K + K^* - K^*K.$$

Since $\widehat{K}$ is a compact operator in X.

Numerical examples

To verify the proposed method, we consider some VIEs, because the exact solution for these problems is available in the literature. For all the examples, the solutions obtained by the proposed method and are thus compared with exact solutions using two polynomial, Laguerre polynomials are trail function and Chebyshev polynomial is the test function. The convergence of each VIEs is calculated by.

$$E = |U_{Ex} - U_{ap}| < \delta$$

where, U_{EX} exact solution and U_{ap} approximation solution.

Example 1 : Consider the following integral equation [1];

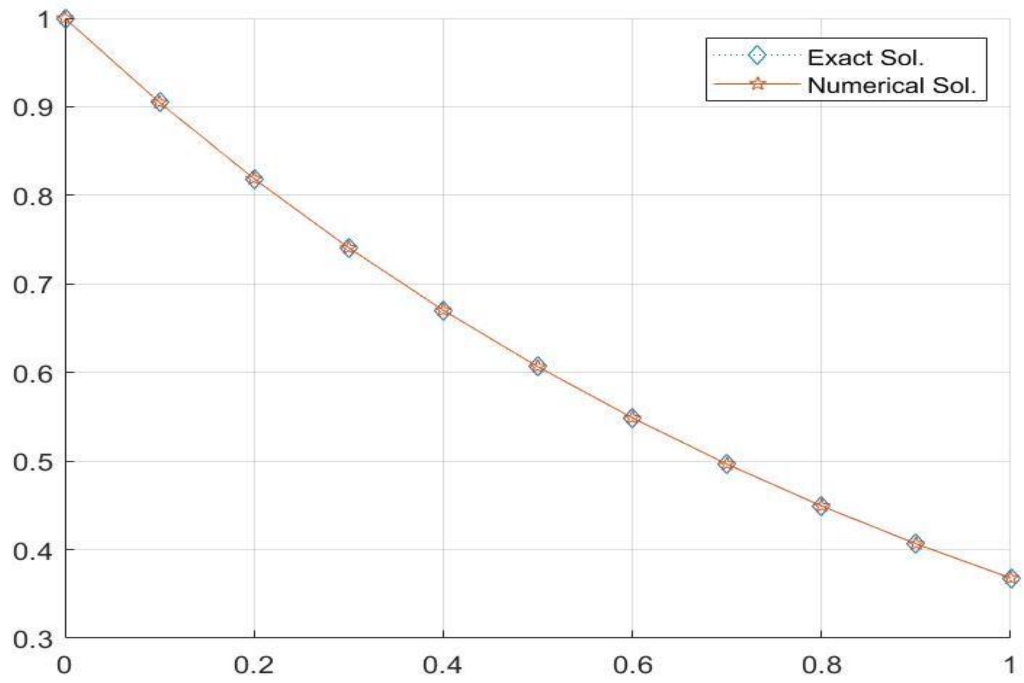
$$\int_0^x e^{x+t}u(t)dt = xe^x,$$

With the exact solution $u(x) = e^{-x}$, for $o \le x \le 1$.

Table 1 Numerical result with analytical solution of **Example 1** for N=10

X	Exact solution	Approximate solution	Absolute Error	Absolute Error [1]
0	1.0000e+00	9.9999e-01	5.9158e-06	1.0416e-02
0.1	9.0484e-01	9.0484e-01	3.6852e-06	1.3148e-02
0.2	8.1873e-01	8.1873e-01	1.2239e-06	1.6630e-03
0.3	7.4082e-01	7.4081e-01	4.3978e-06	1.5820e-03
0.4	6.7032e-01	6.7032e-01	2.0253e-06	9.8100e-03
0.5	6.0653e-01	6.0653e-01	2.8246e-06	6.3180e-03
0.6	5.4881e-01	5.4882e-01	5.0127e-06	7.9750e-03
0.7	4.9659e-01	4.9659e-01	1.4682e-06	1.0090e-03
0.8	4.4933e-01	4.4932e-01	5.2803e-06	9.5900e-04
0.9	4.0657e-01	4.0657e-01	4.1253e-06	5.9500e-03
1	3.6788e-01	3.6791e-01	2.7103e-05	6.7500e-04

L^∞=2.7103e-05.L^2=2.9803e-05.



Example 2 : Consider the following integral equation [1]

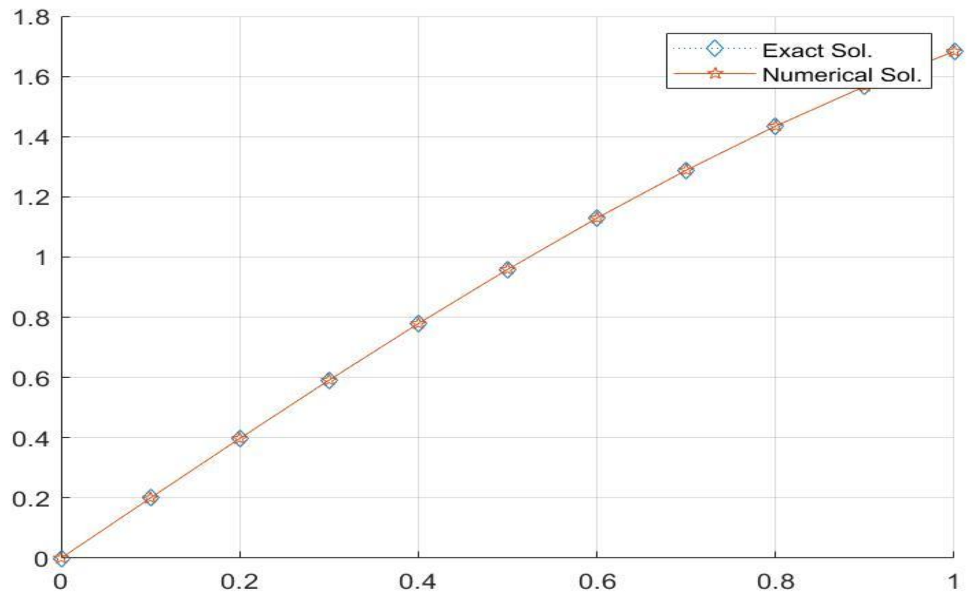
$$\int_0^x \cos(x-t)\,u(t)dt = x\sin x,\ \ 0 \le x \le 1$$

With exact solution $u(x) = 2\sin x$,

Table 2 Numerical result with analytical solution of **Example 2** for N=10

X	Exact solution	Approximate solution	Absolute Error	Absolute Error [1]
0	0	1.2508e-05	1.2508e-05	1.0417e-02
0.1	1.9967e-01	1.9966e-01	8.9861e-06	2.0970e-03
0.2	3.9734e-01	3.9735e-01	9.3692e-06	1.4397e-02
0.3	5.9104e-01	5.9106e-01	1.5180e-05	1.4165e-02
0.4	7.7884e-01	7.7884e-01	1.2156e-07	2.3370e-03
0.5	9.5885e-01	9.5883e-01	1.9479e-05	8.4840e-03
0.6	1.1293e+00	1.1293e+00	2.1473e-05	2.6190e-03
0.7	1.2884e+00	1.2884e+00	4.0628e-06	1.2379e-02
0.8	1.4347e+00	1.4347e+00	3.7726e-05	1.1739e-02
0.9	1.5667e+00	1.5667e+00	1.4824e-05	3.2800e-03
1	1.6829e+00	1.6828e+00	1.8931e-04	4.7800e-02

L^2=1.8931e-04.L^∞=1.9722e-04



Example 3 : Consider the following integral equation [11]

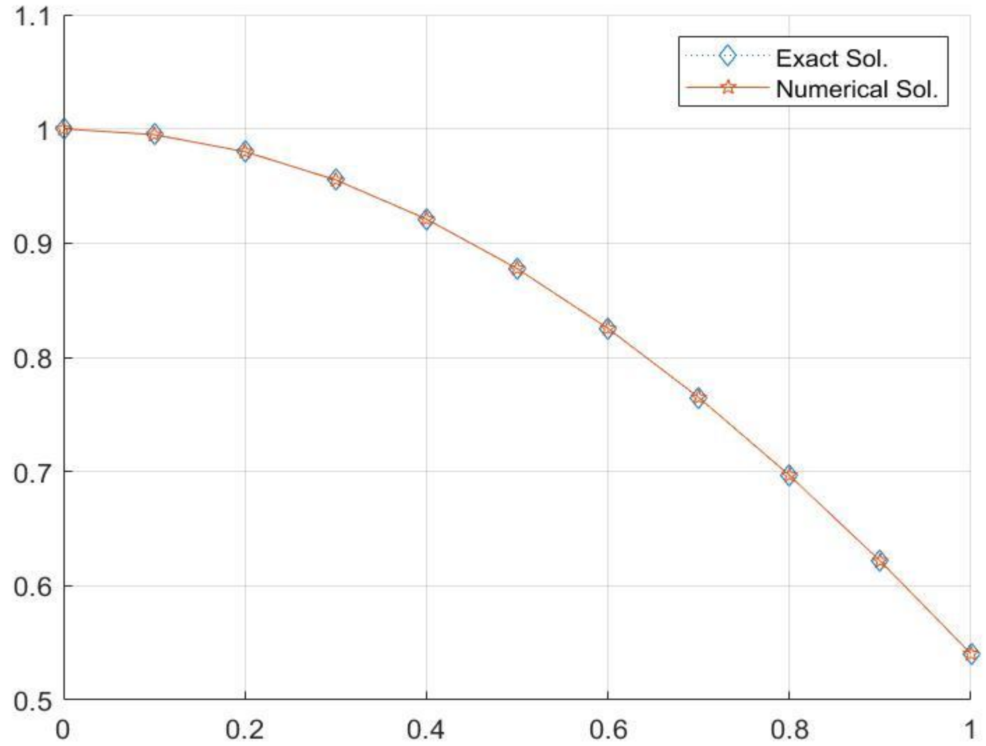
$$\int_0^x (x^2 - t + 2)u(t)dt = (x^2 - x + 2)\sin(x) + 1 - \cos(x); \qquad 0 \le x \le 1$$

With exact solution $u(x) = \cos(x)$,

Table 2 Numerical result with analytical solution of **Example 3** for N=6

X	Exact solution	Approximate solution	Absolute Error	Absolute Error [11]
0	1.0000e+00	1.0001e+00	5.6818e-05	0
0.2	9.5534e-01	9.8005e-01	2.1138e-05	6.7424e-03
0.4	8.7758e-01	9.2108e-01	1.6887e-05	5.1717e-04
0.6	7.6484e-01	8.2534e-01	6.9503e-06	6.4905e-03
0.8	6.2161e-01	6.9668e-01	2.7696e-05	1.5496e-03
1	1.6829e+00	5.4042e-01	1.1321e-04	5.8153e-03

L^2=1.3599e-04.L^∞= 1.1321e-04



Conclusions

We presented the spectral Petrov-Galerkin method (SPGM) in this study, which uses Laguerre polynomials as a trial function and Chebyshev polynomials with a weight function as a test function. In comparison to the cases in [1], and [11] provided in Tables 1-3, the approach was used when N was different. The numerical and exact solutions are quite harmonic, as shown in Figures 1-3, which effectively communicates the numerical solutions. The numerical results are in good agreement with the exact solutions. We evaluated $L^2$and L^∞ norms norms errors to test the accuracy of the suggested method. The method that has been proposed is both effective and reliable.

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