T-weakly Hopfian modules

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Abstract: The concept of Hopfian modules has been extensively studied in the literature. In this paper we introduce the notion of T-weakly Hopfian modules which is a proper generalization of Hopfian modules. We present some properties of these modules. Further, the T-weakly Hopficity of modules over truncated polynomial rings are considered.

Keywords: Hopfian modules, T-weakly Hopfian modules, weakly Hopfian modules.

1. Introduction

Throughout this paper all rings have identity and all modules are unital right modules. The study of modules by properties of their endomorphisms has long been of interest. In [5], Hiremath, introduced the notion of Hopfian modules. A bit later, Varadarajan, introduced the concept of co-Hopfian modules. In [4], a proper generalization of Hopfian modules, called generalized Hopfian modules, was given. A right R-module M is called generalized Hopfian, if any surjective endomorphism of M has a small kernel. In [10], an other proper generalization of Hopfian modules, called weakly Hopfian modules, was given. A right R-module M is called weakly Hopfian, if any small surjective endomorphism of M is an automorphism. In [3], we introduced the concept of μ-Hopfian modules. A right R-module M is called μ-Hopfian, if any surjective endomorphism of M has a μ-small kernel. Such modules and others generalizations were introduced and studied by many authors, (for more information about this and others related topics, see, for instance, [5, 8-10]. Recall that a submodule K of an R-module M is said to be small in M, written K $\ll$ M, if for every submodule L $\leq$ M with K+L=M implies L=M. In [2], a generalization of small submodules, called T-small submodules, was given. A submodule K of an R-module M is said to be T-small in M, written K $\ll_T$ M, with T is a submodule of M, if for every submodule L $\leq$ M such that T $\subseteq$ K+L implies T $\subseteq$ L. If T=M, then K $\ll_T$ M if and only if K $\ll$ M.

By works mentioned we are motivated to introduce in this paper the notion of T-weakly Hopfian modules which is proper generalization of Hopfian modules, and in particular Noetherian modules. We call a module T-weakly Hopfian if any its T-small surjective endomorphism of M is an automorphism. We present some of their properties and examples. Also, we consider the T-weakly Hopfian property of $\frac{M[x]}{(x^{n+1})}$ (as an $\frac{R[x]}{(x^{n+1})}$-module). Varadarajan [9] showed that the left R-module M is Hopfian if and only if the left R[x]-module M[x] is Hopfian if and only if the left $\frac{R[x]}{(x^{n+1})}$-module $\frac{M[x]}{(x^{n+1})}$ is Hopfian, where n is a non-negative integer and x is a commuting indeterminate over R. However, for any nonzero R-module M, the R[x]-module M[x] is never co-Hopfian. In fact, the map "multiplication by x" is injective and non surjective. In [3] we showed that the right $\frac{R[x]}{(x^{n+1})}$ -module $\frac{M[x]}{(x^{n+1})}$ is μ-Hopfian if and only if the right R-module M is μ-Hopfian. We are motivated to prove that, if $\frac{M[x]}{(x^{n+1})}$ is $\frac{T[x]}{(x^{n+1})}$-weakly Hopfian $\frac{R[x]}{(x^{n+1})}$-module then M is T-weakly Hopfian R-module, (Theorem 3.2).

At the end of the paper, some open problems are given.

Let R be a ring and M an R-module. We recall the following definitions and facts:

Definition 1.1.

(1) M is called Hopfian if every surjective endomorphism of M is an automorphism. [5]

(2) M is called co-Hopfian if every injective endomorphism of M is an automorphism. [9]

Remarks

(1) Every Noetherian R-module M (i.e., M has ACC on submodules), is Hopfian .[1]

(1) Every Artinian R-module M (i.e., M has DCC on submodules), is co-Hopfian) . [1]

(1) The additive group $\mathbb{Q}$ of rational numbers is a non-Noetherian non-Artinian $\mathbb{Z}$-module, which is Hopfian and co-Hopfian . [6]

Definition1.2 [2]. Let M and N be modules and T be a submodule of M. An epimorphism f :M ⟶ N is said to be T-small, if Ker f is T-small in M.

Lemma1.3[2]. Let M be an R-module, L ≤ T ≤ M and K ≤ M. Then

(1) If $K \ll_{T} M$, then $K \cap T \ll M$.

(2) $L \ll_{T} M$ if and only if $L \ll T$.

2. Modules whose T-small Surjective Endomorphisms Are Isomorphism

Motivated by the definition of weakly Hopfian modules, we introduce the key definition of this section.

Definition2.1. Let M be an R-module and T be a submodule of M. We say that M is T-weakly Hopfian if any T-small surjective endomorphism of M is an automorphism.

Lemma2.2. For a non-zero module M and T be a submodule of M, the following statements are equivalent.

(i) M is a T-weakly Hopfian module.

(ii) M/K ≅ M for any T-small submodule K ≤ T ≤ M if and only if K = 0.

Proof:

(i) ⇒ (ii) Suppose M ≅M/K for some T-small submodule K ≤ T ≤ M. Let φ: M/K ⟶ M be an isomorphism and

π : M ⟶ M/K the canonical epimorphism. Then the map $\varphi\pi$ is an epimorphism with Ker($\varphi\pi$) = K. Then $\varphi\pi$ is a T-small epimorphism. So $\varphi\pi$ is an isomorphism by (i), and so K=0.

(ii) ⇒ (i) Let f : M ⟶ M be T-small epimorphism. Then M ≅ M/Ker(f) by first isomorphism theorem. From (ii), we get Ker(f)=0. This shows f is an isomorphism. Hence M is T-weakly Hopfian.

Definition2.3. [7]. An R-module M is called quasi-projective if for any surjective homomorphism g of M onto N and any homomorphism, f of M to N, there exists an endomorphism h of M such that: f = gh.

Clearly, every projective module is quasi-projective.

Proposition2.4 Let be an R-module and T be a submodule of M, if M is quasi-projective, then it is T-weakly Hopfian.

Proof: Let M be a quasi-projective module and T ≤ M. Suppose M ≅ M/K for some T-small submodule K ≤ T ≤ M. Let φ: M/K ⟶ M be an isomorphism. The map $\varphi\pi$: M ⟶ M, where π: M ⟶ M/K is a canonical epimorphism has kernel K i.e. Ker($\varphi\pi$) = K. Since M is quasi-projective, there is a g : M ⟶ M such that: $\varphi\pi$ g= 1. Thus, M=Ker($\varphi\pi$) ⊕ Im(g), since Ker($\varphi\pi$)=K $\ll_T$ M, hence by Lemma1.3, Ker($\varphi\pi$) ≪ T then Ker($\varphi\pi$) ≪ M, hence M=Im(g), we must have K=0, and so M is T-weakly Hopfian by Lemma2.2.

The following example show that the class of Hopfian modules form a proper subclass of T-weakly Hopfian modules.

Example An infinite-dimensional vector space is T-weakly Hopfian, but it is not Hopfian, because in [5], a vector space V over a field F is Hopfian if and only if it is finite dimensional.

Recall that a submodule K of an R-module M is said to be essential in M, written K $\leq^e$ M, if for every submodule L ⊆ M with K ∩ L=0 implies L=0. If all non-zero submodules of M are essential in M, then M is called uniform.

Proposition2.5. Let M be a non-singular module and T be a submodule of M, if M is uniform then it is T-weakly Hopfian.

Proof: Suppose M is non-singular uniform R-module. Let f : M ⟶ M be a T-small epimorphism. i.e. Ker f $\ll_T$ M. Suppose Ker f ≠ 0. Then Ker f $\leq^e$ M because M is uniform. So M/Ker f is singular. Since f is an epimorphism, by first isomorphism theorem M/Ker f ≅M. This is impossible because M/Ker f is singular and M is non-singular. Therefore Ker f must be zero. So f is an isomorphism. Hence M is T-weakly Hopfian.

Proposition2.6. Let M be an R-module. If M/N is T-weakly Hopfian for any nonzero T-small submodule N of M, then M itself is T-weakly Hopfian.

Proof: If M is not T-weakly Hopfian. Then there exists a T-small surjection f of M which is not an isomorphism, and f induces an isomorphism g : M/Ker f ⟶ M. If π : M ⟶ M/Ker f denotes the canonical quotient morphism, then π g : M/Ker f ⟶ M/Ker f is a T-small surjection which is not an isomorphism. This is a contradiction.

Proposition2.7. Let M be a quasi-projective module, if M is co-Hopfian then it is T-weakly Hopfian.

Proof: Let f:M ⟶ M be a T-small surjective endomorphism, since M is quasi-projective, then there exists g : M ⟶ M, such that fg=1, then g is a injective endomorphism, since M is co-Hopfian, so g is automorphism, which shows that f is an automorphism, then M is T-weakly Hopfian.

Proposition2.8. Any direct summand of a T-weakly Hopfian module is T-weakly Hopfian.

Proof: Let K be a direct summand of M, there exists N a submodule of M such that M = K ⊕ N. Let f : K ⟶ K be a T-small surjective endomorphism of K, then f induces a surjective endomorphism of M, f$\oplus 1_N$: M ⟶ M with (f$\oplus 1_N$)(k + n) = f (k) + n, where k ∈ K and n ∈ N. Thus Ker (f$\oplus 1_N$)=Ker (f) ⊕ 0$\ll_T$ K ≤ M, then Ker (f$\oplus 1_N$) $\ll_T$ M. Since M is T-weakly Hopfian, f$\oplus 1_N$ is an automorphism of M and hence f is an automorphism of K, then K is T-weakly Hopfian.

The next result gives a condition that a direct sum of two T-weakly Hopfian modules is T-weakly Hopfian.

Proposition2.9. Let M = $M_1 \oplus M_2$ such that M_1, M_2 be invariant submodules under any surjection of M. Then M is T-weakly Hopfian if and only if M_1 and M_2 are T-weakly Hopfian.

Proof

⇒) Clear by Proposition2.8.

⇐) Let f : M ⟶ M be a T-small epimorphism, then f/M_i : M_i ⟶ M_i where i ∈ {1;2}, is a T-small surjection. By assumption, f/M_i is an automorphism. Let f (m_1 +m_2) = 0, then f (m_1)+f(m_2) = 0 and so m_1 = m_2 = 0. Thus f is injective. Then M is T-weakly Hopfian.

3. An analogue to Hilbert Basis Theorem

Let M be an R-module. We will briefly recall the definitions of the modules M[x] and $\frac{\mathrm{M[x]}}{(x^{\mathrm{n}+1})}$ from [8]. The elements of M[x] are formal sums of the form m_0 + $m_1 x$ +...+ $m_k x^k$ with k an integer greater than or equal to 0 and m_i ∈ M. We denote this sum by $\sum_{i=1}^{k} m_i x^i$ ($m_0 x^0$ is to be understood as the element m_0 ∈ M).

Addition is defined by adding the corresponding coefficients. The R[x]-module structure is given by

$$\left(\sum_{i=0}^{k} r_i x^i\right).\left(\sum_{j=0}^{z} m_j x^j\right)= \sum_{l=0}^{k+z} c_l x^l$$

where $c_l = \sum_{i+j=l} r_i m_j$, for any r_i ∈ R, m_j ∈ M.

Any nonzero element β of M[x] can be written uniquely as $\sum_{i=k}^{l} m_i x^i$ with \$l ≥ k ≥ 0, m_i ∈ M, m_k ≠0 and m_l ≠0. In this case, we refer to k as the order of β, l as the degree of β, m_k as the initial coefficient of β, and m_l as the leading coefficient of β.

Let n be any non-negative integer and

$$I_{n+1}= \{0\} \cup \{\ \beta\ ;\ 0 \neq \beta \in \mathrm{R[x]},\ \text{order of } \beta \geq \mathrm{n}+1\backslash\}.$$

Then I_{n+1} is a two-sided ideal of R[x]. The quotient ring R[x]/ I_{n+1} will be called the truncated polynomial ring, truncated at degree n+1. Since R has an identity element, I_{n+1} is the ideal generated by $x^{\mathrm{n}+1}$. Even when R does not have an identity element, we will "symbolically" denote the ring R[x]/ I_{n+1} by $\frac{\mathrm{R[x]}}{(x^{\mathrm{n}+1})}$. Any element of $\frac{\mathrm{R[x]}}{(x^{\mathrm{n}+1})}$ can be uniquely written as ($\sum_{i=0}^{k} r_i x^i$ with r_i ∈ R.

Let

$$D_{n+1}= \{0\} \cup \{\ \beta\ ;\ 0 \neq \beta \in \mathrm{R[x]},\ \text{order of } \beta \geq \mathrm{n}+1\backslash\}.$$

Then D_{n+1} is an R[x]-submodule of M[x]. Since I_{n+1}M[x] ⊂ D_{n+1}, we see that $\frac{R[x]}{(x^{n+1})}$ acts on $\frac{M[x]}{D_{n+1}}$. We denote the module $\frac{M[x]}{D_{n+1}}$ by $\frac{M[x]}{(x^{n+1})}$. The action of $\frac{R[x]}{(x^{n+1})}$ on $\frac{M[x]}{(x^{n+1})}$ is given by

$$(\sum_{i=0}^{n} r_i x^i).(\sum_{j=0}^{n} m_j x^j) = \sum_{l=0}^{n} c_l x^l$$

where $c_l = \sum_{i+j=l} r_i m_j$, for any $r_i \in$ R, $m_j \in$ M.

Any nonzero element β of $\frac{M[x]}{D_{n+1}}$ can be written uniquely as $\sum_{i=k}^{n} m_i x^i$ with \$n ≥ k ≥ 0, $m_i \in$ M, $m_k \neq 0$. In this case, we refer to k as the order of β, m_k as the initial coefficient of β.

The $\frac{R[x_1,\dots,x_k]}{(x_1^{n_1+1},\dots,x_k^{n_k+1})}$-module $\frac{M[x_1,\dots,x_k]}{(x_1^{n_1+1},\dots,x_k^{n_k+1})}$ is defined similarly.

Lemma3.1. Let M be an R-module and T be a submodule of M, if K $\ll_T$ M then $\frac{K[x]}{(x^{n+1})} \ll_{\frac{T[x]}{(x^{n+1})}} \frac{M[x]}{(x^{n+1})}$ as $\frac{R[x]}{(x^{n+1})}$-modules.

Proof: Let $\frac{P[x]}{(x^{n+1})}$ be a submodule of $\frac{M[x]}{(x^{n+1})}$ satisfy $\frac{T[x]}{(x^{n+1})} \subseteq \frac{P[x]}{(x^{n+1})} + \frac{K[x]}{(x^{n+1})}$

For any t ∈ T, we have \$t=$\sum_{i=0}^{n} p_i x^i + \sum_{i=0}^{n} k_i x^i$, where $\sum_{i=0}^{n} p_i x^i \in \frac{P[x]}{(x^{n+1})}$, $\sum_{i=0}^{n} k_i x^i \in \frac{K[x]}{(x^{n+1})}$ and t = $p_0 + k_0$, $p_j = -k_j$, for j = 1, 2, ..., n.

For 0 ≤ l≤n, tx^l=$\sum_{i=l}^{n} p_{i-l} x^i + \sum_{i=l}^{n} k_{i-l} x^i$. Clearly, $\sum_{i=l}^{n} p_{i-l} x^i \in \frac{P[x]}{(x^{n+1})}$. If we define

$$P_l = \{0\} \cup \{\text{initial coefficients of elements of order l in } \frac{P[x]}{(x^{n+1})}\}$$

then each P_l is an R-module, and $P_0 \subseteq P_1 \subseteq \dots \subseteq P_n$. We have T ⊆ P_0 + K, since K $\ll_T$ M, then T ⊆ P_0. Consequently, T ⊆ P_i , for i = 0, 1, ..., n, and it follows easily that $\frac{T[x]}{(x^{n+1})} \subseteq \frac{P[x]}{(x^{n+1})}$.

Theorem3.2. Let M be a module. If $\frac{M[x]}{(x^{n+1})}$ is $\frac{T[x]}{(x^{n+1})}-$weakly Hopfian $\frac{R[x]}{(x^{n+1})}-$module, then M is T-weakly Hopfian R-module.

Proof: Let f : M ⟶ M be any T-small epimorphism in R-module, then g : $\frac{M[x]}{(x^{n+1})} \longrightarrow \frac{M[x]}{(x^{n+1})}$ defined by g($\sum_{i=0}^{n} a_i x^i$)= $\sum_{i=0}^{n} f(a_i) x^i$is a surjective endomorphism in $\frac{R[x]}{(x^{n+1})}$-module and Ker g=$\frac{\text{Ker (f)}[x]}{(x^{n+1})}$. Since Ker f is \$T\$-small in M then by Lemma3.1. Ker g=$\frac{\text{Ker (f)}[x]}{(x^{n+1})}$ is $\frac{T[x]}{(x^{n+1})}$-small in $\frac{M[x]}{(x^{n+1})}$. Since $\frac{M[x]}{(x^{n+1})}$ is $\frac{T[x]}{(x^{n+1})}$-weakly Hopfian $\frac{R[x]}{(x^{n+1})}$-module, hence g is an automorphism in $\frac{M[x]}{(x^{n+1})}$. Therefore f is an automorphism in M, and finally M is T-weakly Hopfian.

Corollary3.3. Let M be a module. If \$$\frac{M[x_1,\dots,x_k]}{(x_1^{n_1+1},\dots,x_k^{n_k+1})}$ is $\frac{T[x_1,\dots,x_k]}{(x_1^{n_1+1},\dots,x_k^{n_k+1})}$-weakly Hopfian $\frac{R[x_1,\dots,x_k]}{(x_1^{n_1+1},\dots,x_k^{n_k+1})}$-module, then M is T-weakly Hopfian R-module.

Proof: Use induction and the

$\left(\frac{R[x_1,\ldots,x_{k-1}]}{(x_1^{n_1+1},\ldots,x_{k-1}^{n_{k-1}+1})}\right)\frac{[x_k]}{(x_k^{n_k+1})}$-module isomorphism $\left(\frac{M[x_1,\ldots,x_{k-1}]}{(x_1^{n_1+1},\ldots,x_{k-1}^{n_{k-1}+1})}\right)\frac{[x_k]}{(x_k^{n_k+1})} \simeq \frac{M[x_1,\ldots,x_k]}{(x_1^{n_1+1},\ldots,x_k^{n_k+1})}$

and ring isomorphism $\left(\frac{R[x_1,\ldots,x_{k-1}]}{(x_1^{n_1+1},\ldots,x_{k-1}^{n_{k-1}+1})}\right)\frac{[x_k]}{(x_k^{n_k+1})} \simeq \frac{R[x_1,\ldots,x_k]}{(x_1^{n_1+1},\ldots,x_k^{n_k+1})}$

Open Problems

(1) What is the structure of rings whose finitely generated right modules are T-weakly Hopfian?

(2) Let R be a ring with identity, and M be a T-weakly Hopfian module. Is $M[X, X^{-1}]$ $T[X, X^{-1}]$-weakly Hopfian in $R[X, X^{-1}]$-module?

(3) Let R be a T-weakly Hopfian ring and $n \geq 1$ an integer. Is the matrix ring M_n (R) T-weakly Hopfian?

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References

[1] Anderson, F.W. and Fuller, K.R. Rings and Categories of Modules, Grad. Texts in Math. Springer-Verlag, New York. vol. 13, (1992).

[2] Beyranvand, R. And Moradi, F. Small submodules with respect to an arbitrary submodule. Journal of Algebra and Related Topics, 3(2), (2015) 43--51.

[3] El Moussaouy, A., Ziane, M.: Extension property of automorphism of modules. Journal of Mathematical Sciences : Advances and Applications 67, 61-71 (2021).

[4] El Moussaouy, A., Ziane, M.: Modules in which every surjective endomorphism has a μ-small kernel. Ann Univ Ferrara 66, 325-337 (2020).

[5] Ghorbani, A. and Haghany, A. Generalized Hopfian modules. J. Algebra. 255(2) (2002), 324--341.

[6] Hiremath, V.A. Hopfian rings and Hopfian modules, Indian J. Pure Appl.Math. 17 , no. 7 (1986), 895--900.

[7] Hmaimou, A. Kaidi, A. and Sanchez Campos, E. Generalized Fitting modules and rings. J. Algebra. 308 (2007), 199--214.

[8] Koehler, A.: Quasi-projective and quasi-injective modules. Pacific J. Math. 36(3), 713-720 (1971).

[9] Varadarajan, K. A generalization of Hilbert's basis theorem. Comm. Algebra 10(20) (1982), 2191--2204.

[10] Varadarajan, K. Hopfian and co-Hopfian objects. Publications Matematiques. 36 (1992), 293--317.

[11] Wang, Y. Generalizations of Hopfian and co-Hopfian modules. Int. J. Math. Sci. 9 (2005), 1455--1460.