Riesz Summability of Factored Infinite Series Including Sequence and Fourier series

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Abstract: We established a new theorem under the new condition is known as the theorem of Mazhar and found its application to Fourier series.

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1. **Introduction.** Works on the absolute sum factorization of infinite series and Fourier series is the absolute sum factorization of infinite series and Fourier series is done by several reseachers {[1,2,3…15]}.In[9], it proved the following theorems of which deals by $\left|\overline{N}, g_n\right|_k$ additive sum of infinite series and factors.
2. **Definitions and Notations.** A sequence (A_n) is said to be an almost positive increasing sequence if there exists a positive increasing sequence (b_n) and two positive constants R and S respectvely . Such that $R\,b_n \le A_n \le S\,b_n$,by [1]. The sequence (μ_n), we write that $\Delta\mu_n = \mu_n - \mu_{n+1}$ and$\Delta^2\mu_n = \Delta\mu_n - \Delta\mu_{n+1}$,where (μ_n) is called bounded variation, if $\sum_{n-1}^{\infty}|\Delta\mu_n| < \infty$. Let$\sum_{n=0}^{\infty}\alpha_n$ with the partial sums(β_n). Here q_n^α be the nth order of Cesàro means ω, $\omega > -1, \omega > -1$, the sequence (ne_n),by [6],

$$q_n^\alpha = \frac{1}{B_n^\alpha}\sum_{i=1}^{n} A_{n-i}^{\alpha-i}\, i\, a_i\,, \qquad \left(q_n^1 = q_n\right) \tag{2.1}$$

where

$$B_n^\alpha = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!} = O\left(n^\alpha\right), \quad B_n^\alpha = 0 \qquad \text{for } n > 0. \tag{2.2}$$

The series $\sum \alpha_n$ is said to be summable $|C,w|_k$, $k \ge 1$,, {see [7]}

$$\sum_{n=1}^{\infty}\frac{1}{n}\left|q_n^\omega\right| < \infty \tag{2.3}$$

If we take $\omega = 1$ then $\left|C,\omega\right|_k$ it becomes to the form $\left|C,1\right|_k$, which is the form of summability. Now (g_n) be a sequence of +ve real numbers is such that

$$G_n = \sum_{i=0}^{n} g_i \to \infty \text{ as } n \to \infty, \qquad (G_{-i} = g_{-i} = 0, i \geq 1) \tag{2.4}$$

with sequences transformation such that

$(\beta_n) \to (\omega_n)$

and $$\omega_n = \frac{1}{G_i} \sum_{i=0}^{n} g_i \ \beta_i \tag{2.5}$$

defines the sequence (ω_n) of Riesz mean or $(\overline{N}, g_n)$ means of the sequence (β_n), and its generated coefficient of sequence (g_n) { see[8] }.

The series $\sum \alpha_n$ is said to be summable $\left|\overline{N}, g_n\right|_k, k \geq 1,$ if {see [3]}

$$\sum_{n=1}^{\infty} \left(\frac{G_n}{g_n}\right)^{k-1} \left|\omega_n - \omega_{n-1}\right|^k < \infty.$$

In the special case when $g_n = 1$ for all n (respectively. $k = 1$), $\left|\overline{N}, g_n\right|_k$, summability is the same as $\left|C,1\right|_k$, (resp. $\left|\overline{N}, g_n\right|_k$ {see [10]}) summability. Also if we take $g_n = \frac{1}{n+1}$ and $k = 1,$ then we obtained $\left|J, \log n, 1\right|$ summability {see [2]}.

Let f be periodic function with period 2π and Lebesgue integrable over $(-\pi, \pi)$. The trigonometric Fourier series of f is

$$f \sim \frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx\right] + \sum_{m=1}^{\infty}\left[\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx\right)\cos mx + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx\right)\sin mx\right]$$

3. **Known theorem[9]**

Let (A_n) be an almost increasing sequence. If the sequence (A_n), (μ_n) and (g_n) satisfy the conditions

$$\left|\mu_m\right| A_n = O(1) \quad , m \to \infty \tag{3.1}$$

$$\sum_{n=1}^{m} nA_n \left|\Delta^2 \mu_n\right| = O(1) \quad , m \to \infty \tag{3.2}$$

$$\sum_{n=1}^{m} \frac{G_n}{n} = O(G_m) \quad , m \to \infty \tag{3.3}$$

$$\sum_{n=1}^{m} \frac{g_n}{G_n} \left|z_n\right|^k = O(A_m) \quad , m \to \infty \tag{3.4}$$

$$\sum_{n=1}^{m} \frac{\left|z_n\right|^k}{n} = O(A_m) \quad , m \to \infty \tag{3.5}$$

Then the series $\sum \alpha_n \mu_n$ is summable $\left|\overline{N}, g_n\right|_k, k \geq 1.$

4. **Main theorem.** The main aim of this paper is to improve known theorem result under new condition. Now we shall prove

Theorem: Let (A_n) be an almost increasing sequence. If the (A_n), (μ_n) and (g_n) satisfy the condition (3.1), (3.2), (3.3) and

$$\sum_{n=1}^{m} \frac{g_n}{G_n} \frac{|z_n|^k}{A_n^{k-1}} = O(A_m) \quad , m \to \infty \tag{4.1}$$

$$\sum_{n=1}^{m} \frac{|z_n|^k}{nA_n^{k-1}} = O(A_m) \quad , m \to \infty \tag{4.2}$$

Hence the summable $\sum \alpha_n \mu_n$ is $\left|\overline{N}, g_n\right|_k, k \geq 1.$

Note: Condition (4.1) is converted to condition (3.4), when $k = 1$, $k > 1$, condition (4.1),which is weaker condition of (3.4), but the converse is not true. The fact, if (3.4) is verified, we get

$$\sum_{n=1}^{m} \frac{g_n}{G_n} \frac{|z_n|^k}{A_n^{k-1}} = O\left(\frac{1}{A_1^{k-1}}\right) \sum_{n=1}^{m} \frac{g_n}{G_n} |z_n|^k = O(1) \sum_{n=1}^{m} \frac{g_n}{G_n} |t_n|^k = O(A_m) \quad .$$

When $k \geq 1,$ the following example is sufficient case but the converse is false.

Let $A_n = n^{\delta}, 0 < \delta < 1,$ and then construct a sequence (h_n) such that

$$h_n = \frac{g_n}{G_n} \frac{|z_n|^k}{A_n^{k-1}} = A_n - A_{n-1} ,$$

Hence

$$\sum_{n=1}^{m} \frac{g_n}{G_n} \frac{|z_n|^k}{A_n^{k-1}} = \sum_{n=1}^{m} (A_n - A_{n-1}) = A_m = m^{\delta} ,$$

Therefore

$$\sum_{n=1}^{m} \frac{g_n}{G_n} |z_n|^k = \sum_{n=1}^{m} (A_n - A_{n-1}) A_m^{k-1}$$

$$= \sum_{n=1}^{m} \left(n^{\delta} - (n-1)^{\delta}\right) n^{\delta(k-1)}$$

$$\geq \delta \sum_{n=1}^{m} n^{\delta}\, n^{\delta(k-1)} = \delta \sum_{n=1}^{m} n^{\delta k-1)} \sim \frac{m^{\delta k}}{k} \quad \text{as} \quad m \to \infty$$

Its Proved that

$$\frac{1}{A_m}\sum_{n=1}^{m}\frac{g_n}{G_n}\left|z_n\right|^k \to \infty, \qquad \text{as } m \to \infty ,$$

to be provided $k>1$. This shows that (3.4) implies (4.1) but converse is not true. If we putt $g_n = 1$ for all n, we get the identical results of (4.2) and (3.5). The following lemma is necessary for the proof of our theorem.

Lemma [9]: we have

$$\sum_{n=1}^{m} A_n \left|\Delta\mu_n\right| < \infty \quad \text{,under the condition of the theorem} \tag{4.3}$$

$$nA_n \left|\Delta\mu_n\right| = O(1) \quad \text{as} \quad n \to \infty \tag{4.4}$$

5. Proof of the main theorem.

Let (L_n) be the sequence of $(\overline{N}, g_n)$ means of thr series $\sum \alpha_n \mu_n$. Then we have

$$L_n = \frac{1}{G_n}\sum_{\nu=0}^{n} g_\nu \sum_{r=0}^{\nu} \alpha_r \mu_r$$

$$= \frac{1}{g_n}\sum_{i=0}^{n}\left(g_i - g_{i-1}\right)\alpha_r \mu_r \tag{5.1}$$

Then, for $n \geq 1$, we get

$$L_n - L_{n-1} = \frac{g_n}{G_n G_{n-1}}\sum_{i=0}^{n}\frac{g_{i-1}\mu_i}{i} \; i\alpha_i \tag{5.2}$$

Applying Abel's transformation to the right hand side of (5.2), we have

$$L_n - L_{n-1} = \frac{g_n}{G_n G_{n-1}}\sum_{i=0}^{n-1}\Delta\left(\frac{g_{i-1}\mu_i}{i}\right)\sum_{r=1}^{i} r\, a_r + \frac{g_n \mu_n}{nG_n}\sum_{i=1}^{n} n\, a_i$$

$$= \frac{(n+1)g_n z_n \mu_n}{nG_n} - \frac{g_n}{G_n G_{n-1}}\sum_{i=0}^{n-1} g_i z_i \mu_i \frac{i+1}{i}$$

$$+ \frac{g_n}{G_n G_{n-1}}\sum_{i=1}^{n-1} g_i \Delta\mu_i z_i \frac{i+1}{i} + \frac{g_n}{G_n G_{n-1}}\sum_{i=1}^{n-1} g_i \Delta \mu_i z_i . \frac{1}{i}$$

$$= L_{n,1} + L_{n,2} + L_{n,3} + L_{n,4}$$

To prove that by Minkowski's inequality,

$$\sum_{n=1}^{\infty}\left(\frac{G_n}{g_n}\right)^{k=1}\left|L_{n,r}\right|^k<\infty \quad \text{, for } r=1,2,3,4.$$

We have by applying Abel's transforms,

$$\sum_{n=1}^{\infty}\left(\frac{G_n}{g_n}\right)^{k=1}\left|L_{n,1}\right|^k=O(1)\sum_{n=1}^{m}\left|\mu_n\right|^{k-1}\left|\mu_n\right|\frac{g_n}{G_n}\left|z_n\right|^k$$

$$=O(1)\sum_{n=1}^{m}\left|\mu_n\right|\frac{g_n}{G_n}\frac{\left|z_n\right|^k}{A_v^{k-1}}$$

$$=O(1)\sum_{n=1}^{m}\Delta\left|\mu_n\right|\sum_{i=1}^{n}\frac{g_i}{G_i}\frac{\left|z_i\right|^k}{A_i^{k-1}}+O(1)\left|\mu_m\right|\sum_{n=1}^{m}\frac{g_n}{G_n}\frac{\left|z_n\right|^k}{A_n^{k-1}}$$

$$=O(1)\sum_{n=1}^{m-1}\Delta\left|\mu_n\right|A_n+O(1)\left|\mu_m\right|A_m$$

$$=O(1) \quad , m\to\infty,$$

By the hypothesis of Lemma and theorem . Similarly $L_{n,1}$, we have

$$\sum_{n=2}^{m=1}\left(\frac{G_n}{g_n}\right)^{k=1}\left|L_{n,2}\right|^k=O(1)\sum_{n=2}^{m+1}\frac{g_n}{G_n\,G_{n-1}}\left(\sum_{i=1}^{n-1}g_i\left|z_i\right|^k\left|\mu_i\right|^k\right)\left(\frac{1}{G_{n-1}}\sum_{i=1}^{n-1}g_i\right)^{k-1}$$

$$=O(1)\sum_{i=1}^{m}\left|\mu_i\right|^{k-1}\mu_i\left|z_k\right|^k\sum_{n=i+1}^{m+1}\frac{g_n}{G_n\,G_{n-1}}$$

$$=O(1)\sum_{i=1}^{m}\left|\mu_i\right|\frac{g_i}{G_i}\frac{\left|z_i\right|^k}{A_i^{k-1}}$$

$$=O(1) \quad \text{as} \quad m\to\infty,$$

Next, by using (3.3), we get that

$$\sum_{n=2}^{m+1}\left(\frac{G_n}{g_n}\right)^{k=1}\left|L_{n,3}\right|^k=O(1)\sum_{n=2}^{m+1}\frac{g_n}{G_n\,G_{n=1}^k}\left(\sum_{i=1}^{n-1}g_i\left|\Delta\mu_i\right|\left|z_i\right|\right)^k$$

$$=O(1)\sum_{n=2}^{m+1}\frac{g_n}{G_n\,G_{n=1}^k}\left(\sum_{i=1}^{n-1}\frac{G_i}{i}\,i\left|\Delta\mu_v\right|\left|z_i\right|\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{g_n}{G_n\,G_{n-1}}\left(\sum_{i=1}^{n-1}\frac{G_i}{i}\,\left(i\left|\Delta\mu_i\right|\right)^k\ \left|z_i\right|^k\right)\left(\frac{1}{G_{n-1}}\sum_{i=1}^{n-1}\frac{G_i}{i}\right)^{k-1}$$

$$= O(1)\sum_{i=1}^{m}\frac{G_i}{i}\left(i\left|\Delta\mu_i\right|\right)^{k-1} i\left|\Delta\mu_i\right|\left|z_i\right|^k\ \sum_{n=i+1}^{m+1}\frac{g_n}{G_n\,G_{n-1}}$$

$$= O(1)\sum_{i=1}^{m}\left(i\left|\Delta\mu_i\right|\right)\ \frac{\left|z_i\right|^k}{i\,A_i^{k-1}}$$

$$= O(1)\sum_{i=1}^{m}\Delta\left(i\left|\Delta\mu_i\right|\right)\sum_{r=1}^{i}\frac{\left|z_r\right|^k}{r\,A_r^{k-1}} + O(1)\,m\left|\Delta\mu_m\right|\sum_{i=1}^{m}\frac{\left|z_i\right|^k}{i\,A_i^{k-1}}$$

$$= O(1)\sum_{i=1}^{m-1}\Delta\left(i\left|\Delta\mu_i\right|\right)A_i + O(1)\,m\left|\Delta\mu_m\right|A_m$$

$$= O(1)\sum_{i=1}^{m-1}\left(iA_i\left|\Delta^2\mu_v\right|\right) + O(1)\sum_{i=1}^{m-1}A_i\left|\Delta\mu_v\right| + O(1)\ m\left|\Delta\mu_m\right|A_m$$

$$= O(1)\ \text{, as } m\to\infty\text{,}$$

By the hypothesis of theorem and Lemma. Lastly, by using (3.3) as in $L-(n,1)$ we have that

$$\sum_{n=2}^{\infty}\left(\frac{G_n}{g_n}\right)^{k=1}\left|L_{n,4}\right|^k \le \sum_{n=2}^{m+1}\frac{g_n}{G_n\,G_{n-1}^k}\left(\sum_{i=1}^{n-1}\frac{g_i}{i}\left|\mu_{i+1}\right|^k\left|z_i\right|^k\right)$$

$$= O(1)\sum_{n=2}^{m+1}\frac{g_n}{G_n\,G_{n-1}}\left(\sum_{i=1}^{n-1}\frac{g_i}{i}\left|\mu_{i+1}\right|^k\left|z_i\right|^k\right)\left(\frac{1}{G_{n-1}}\sum_{i=1}^{n-1}\frac{G_i}{i}\right)^{k-1}$$

$$= O(1)\sum_{i=1}^{m}\frac{G_i}{i}\left|\mu_{i+1}\right|^{k-1}\left|\mu_{i+1}\right|^k\sum_{n=i+1}^{m+1}\frac{g_n}{G_n\,G_{n-1}}$$

$$= O(1)\sum_{i}^{m}\left|\mu_{i+1}\right|\frac{\left|z_i\right|^k}{i\,A_i^{k-1}} = O(1)\quad ,\ m\to\infty$$ .Which is the proof of the theorem.

Remarks:

If we put $g_n = 1$ for all n, we get a new result with $\left|C,1\right|_k$ with additive of an infinite series factors. Furthermore, if we put $k=1$, we get a new result $\left|\overline{N}, g_n\right|$ to the additive of an infinite series with factors. Finally, if we put $g_n = \dfrac{1}{n+1}$, k=1 we get a new result $\left|J,\log n,\ 1\right|$ the additive of the infinite series with factorized.

Declaration of competitive interest.

The author declares that they have no competing interests

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