Parameters Estimation of Lindely Distribution a Comparative Study

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Abstract: In this paper, we study the three-parameter Lindley distribution (LD) and its mathematical and statistical properties such as the moment generating function, quantile function, reliability function and others. And we estimate the parameters of this distribution by simulation study in eight methods; the maximum Likelihood (ML), ordinary least squares (OLS), method of weighted least squares (WLS), percentiles method (P.C), method of maximizing distance multiplication (MDM), Anderson's Darling (AD), Anderson from the right (RAD) and Cramer-von Mess (C-VM) method. According to the simulation study for three types of samples, the maximum likelihood method was the best method, followed closed the Anderson Darling method, then the method of maximizing the multiplication of distances, we use the statistical criteria (BIAS, MSE, MRE), for comparison between the methods.

Keywords: Lindley distribution, Cramer-von Mess. Maximum Likelihood, maximizing distance multiplication, Anderson Darling, Anderson Darling from the right, least squares method, weighted least squares, percentiles method.

1. Introduction

The one-parameter Lindley distribution is introduced in 1958 by the English statistician Dennis Victor Lindley (1923 - 2013) , he suggested the p.d.f is as following:

$$\boldsymbol{f(z;\mu)=\frac{\mu^2}{\mu+1}(1+z)e^{-\mu z}}, \qquad \boldsymbol{z}>0\ ,\ \boldsymbol{\mu}>0 \qquad (1)$$

μ is thescale parameterwhere

the two-parameter Lindley distribution proposed by Shanker and Misher 2013

$$\boldsymbol{f(z;\mu,\sigma)=\frac{\mu^2}{\sigma\mu+1}(\sigma+z)e^{-\mu z}}, \quad \boldsymbol{z}>0\ ,\boldsymbol{\mu}>0\ ,\ \sigma\boldsymbol{\mu}>-1 \qquad (2)$$

2.The three – parameter Lindley distribution [9]

The probability density function (pdf) is as following:

$$f(z_i;\mu,\sigma,\rho)=\frac{\mu^2}{\mu\sigma+\rho}(\sigma+\rho z)e^{-\mu z}, \qquad z>0\,,\mu>0\,,\sigma>0\,,\rho>0 \qquad (3)$$

we can verified that the one parameter Lindley distribution and the two-parameter Lindley distribution (with some others versions) is particular cases of this distribution .The importance of this distribution that used for modeling lifetime data , in many fields. which suggested by Shanker et. al. In 2017.

The shape of pdf of the LD has follow a unimodal shape function ,right –skewed or reversed –j shaped as shown in figure (1)

And the cumulative distribution (cdf) is in the following form*:*

$$F\ (z;\mu,\sigma,\rho)=1-[1+\frac{\mu\rho z}{\mu\sigma+1}]\,e^{-\mu z}\ , \qquad z>0\ \ ,\mu\ >0\ \ ,\rho>0\ \ ,\mu>0 \qquad (4)$$

It is a non-decreasing monotonic function as in the Figure (2)

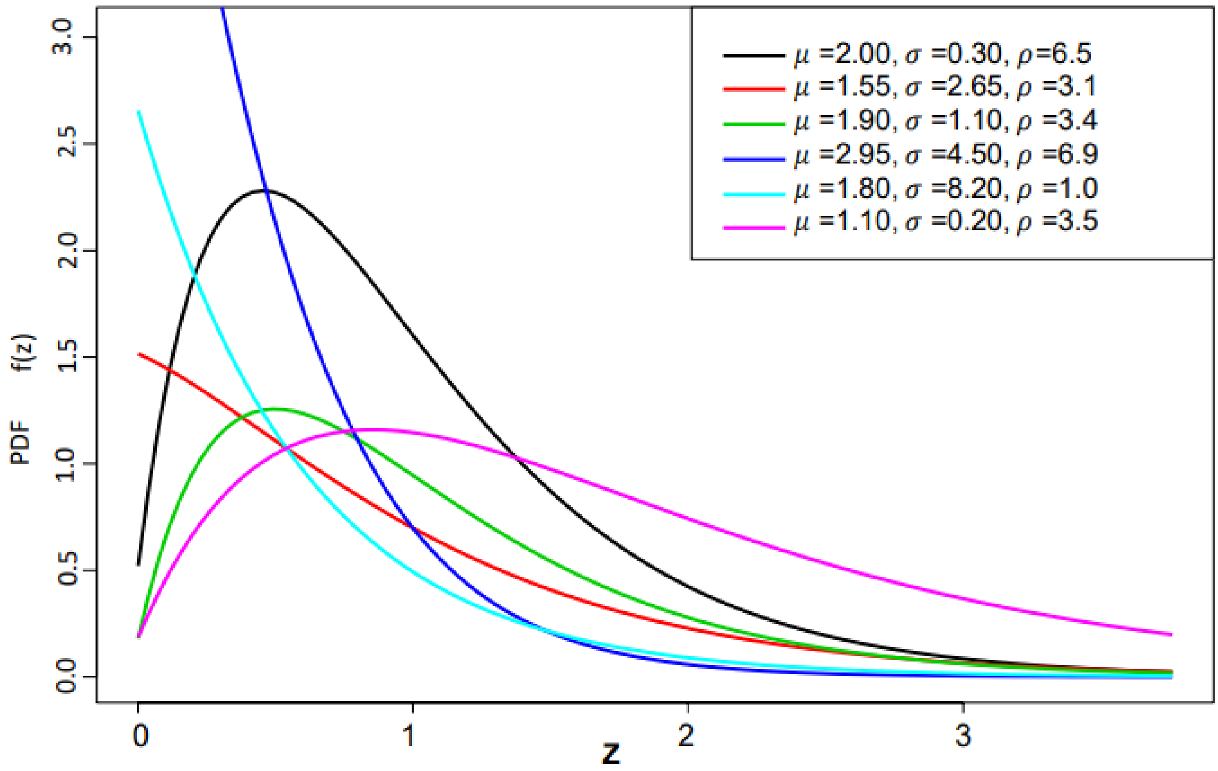

Figure (1) Graph of the pdf for LD for different μ , σ , ρ

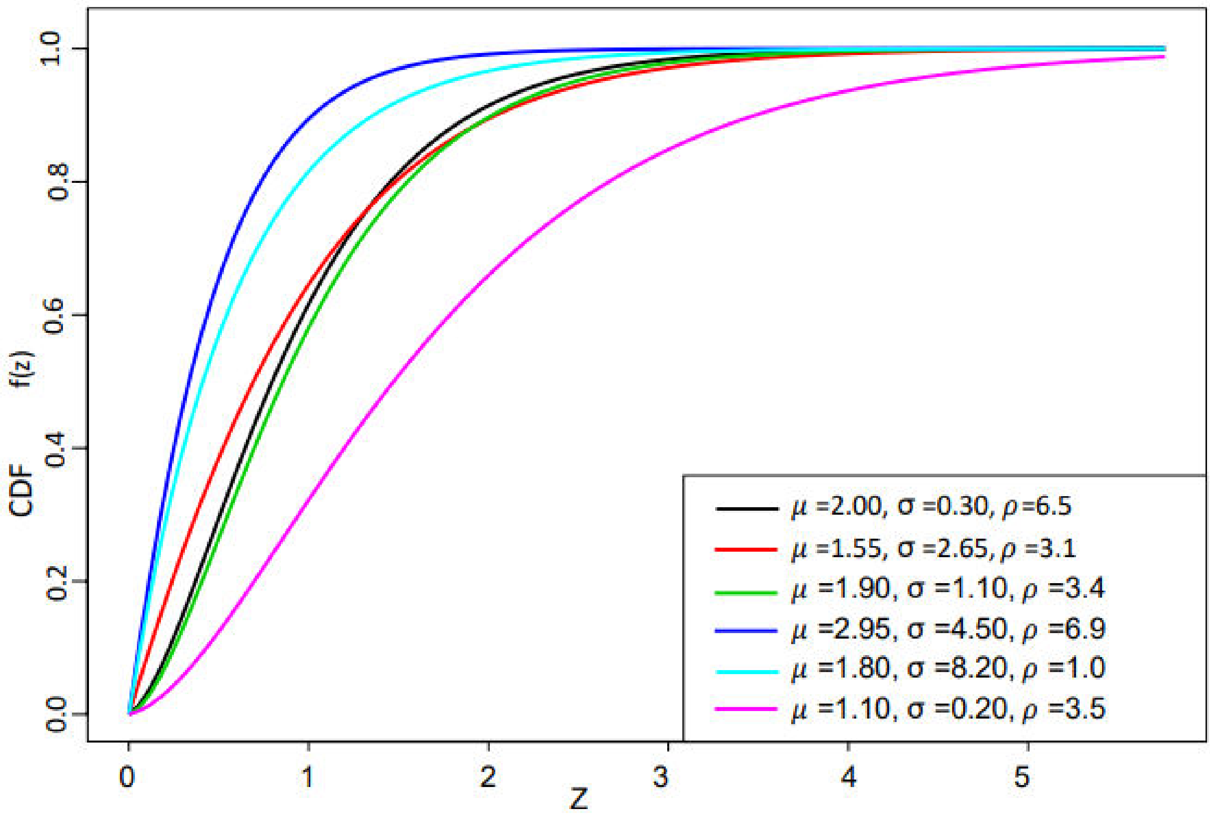

Figure (2) Graph of the cdf for LD for different μ , σ , ρ

The hazard rate function also known as the failure rate function has the form

$$h(z) = \frac{f(z)}{1 - F(z)} = \frac{\frac{\mu^2}{\mu\sigma+\rho}(\sigma + \rho z)e^{-\mu z}}{1 - \left[1 + \frac{\mu\rho z}{\mu\sigma+1}\right] e^{-\mu z}} \quad (5)$$

The rest of this paper is ordered as follows :in section 2,we discuss some of the mathematical properties of the LD distribution .in section 3,we obtain and estimate the three parameters by the methods (MLE, OLSE,WLSE,P.C,CVME, MPSE, RADE,ADE) . in section 4,we introduced simulation study.in section 5, the conclusion.

3.Some Propertiesof Lindley Distribution (LD)

The distribution of (LD) some other properties in addition to the properties introduce by [4] Rama Shanker such as the quantile function (Qf), the moment generating function (m.g.f) and the reliability function R(z)

3.1.Quantile function:

Thequantile function Qf is used to estimate and generate random numbers [5] when distributing important functions because of their It has many uses in statistical applications. It is explained as follows:

Let $u = F(z)$, then $u = 1 - (1 + \frac{\mu\rho z}{\mu\sigma+\rho})e^{-\mu z}$

$$u - 1 = -(1 + \frac{\mu\rho z}{\mu\sigma + \rho})e^{-\mu z} \quad (6)$$

let $y = 1 + \frac{\mu\rho z}{\mu\sigma+\rho}$, then $y - 1 = \frac{\mu\rho z}{\mu\sigma+\rho}$

$$(y - 1)(\mu\sigma + \rho) = \mu\rho z$$

$z = \frac{(Y-1)(\mu\sigma+\rho)}{\mu\rho}$

we substitute the value of z in to the equation (6)

$$u - 1 = -(1 + \frac{(y-1)(\mu\sigma+\rho)}{\mu\sigma+\rho})\, e^{-\frac{(y-1)(\mu\sigma+\rho)}{\rho}}$$

$$w(u)e^{w(u)} = w(u)$$

where w is the Lambert function, and after some steps we can find that

$$\therefore z = \frac{-\rho w\left(\frac{(u-1)(\mu\sigma+\rho)}{\rho e^{\frac{\mu\sigma+\rho}{\rho}}}\right) + \mu\sigma + \rho}{\mu\rho} \quad (7)$$

where $u \sim U(0,1)$

3.2.The moment generating function:

$$M_z(t) = \int_0^\infty e^{tz} f(z)dz = \int_0^\infty e^{tz} \frac{\mu^2}{\mu\sigma+\rho}(\sigma + \rho z)e^{-\mu z}dz$$

$$M_z(t) = \frac{\mu^2}{\mu\sigma+\rho}\int_0^\infty e^{-(\mu-t)z}(\sigma + \rho z)dz$$

Let $Y = (\mu - t)\, z$ then $z = \frac{y}{\mu-t}$, $dz = \frac{dy}{\mu-t}$

$$M_z(t) = \frac{\mu^2}{\mu\sigma+\rho}\int_0^\infty e^{-y}\left(\sigma + \rho\left(\frac{y}{\mu-t}\right)\right)\frac{dy}{\mu-t}$$

$$M_z(t) = \frac{\mu^2}{\mu\sigma+\rho}[\int_0^\infty \sigma e^{-y}\frac{dy}{\mu-t} + \int_0^\infty \frac{\rho}{(\mu-t)^2}\, ye^{-y}dy]$$

$$M_z(t) = \frac{\mu^2}{\mu\sigma+\rho}[\frac{\sigma}{\mu-t}\int_0^\infty e^{-y}dy + \frac{\rho}{(\mu-t)^2}\int_0^\infty ye^{-y}dy]$$

$$M_z(t) = \frac{\mu^2}{\mu\sigma+\rho}[\frac{-\sigma}{\mu-t}e^{-y} + \frac{\rho}{(\mu-t)^2}\gamma_2]_0^\infty$$

$$M_z(t) = \frac{\mu^2\sigma(\mu-t)+\mu^2\rho}{(\mu\sigma+\rho)(\mu-t)^2} \qquad (8)$$

3.3.Reliability function:

The reliability function formula is as following:

$R\ (z;\sigma,\mu,\rho) = 1 - F\ (z;\sigma,\mu,\rho = 1 - (1 - \left[1+\frac{\mu\rho z}{\mu\sigma+\rho}\right]e^{-\mu z})$

$$= \frac{\mu\sigma+\rho+\mu\rho z}{\mu\sigma+\rho}e^{-\mu z}, \qquad z>0\ ,\mu>0\ ,\sigma>0\ ,\rho>0 \qquad (9)$$

4.ParametersEstimation

In this section, we review the method for estimating the parameters of the Lindley probability distribution in order to show the best method for estimating the parameters. We dealt with eight compact methods and know for estimate. These methods are as follows:

4.1.Maximum Likelihood (ML).

The likelihood function, L of the three parameter Lindley distribution (3) is given by

$L(\sigma,\mu,\rho;z\ ) = (\frac{\mu^2}{\mu\sigma+\rho})\prod_{i=1}^n(\sigma+\rho z_i)e^{-n\mu z}$

$$\frac{\partial lnL}{\partial\mu} = \frac{2n}{\mu} - \frac{n\sigma}{\mu\sigma+\rho} - n^2\bar{z} = 0 \qquad (10)$$

$$\frac{\partial lnL}{\partial\sigma} = \frac{-n\mu}{\mu\sigma+\rho} + \sum_{i=1}^n \frac{1}{\sigma+\rho z_i} = 0 \qquad (11)$$

$$\frac{\partial lnL}{\partial\rho} = \frac{-n}{\mu\sigma+\rho} + \sum_{i=1}^n \frac{z_i}{\sigma+\rho z_i} = 0 \qquad (12)$$

The second derivatives is:

$$\frac{\partial^2 lnL}{\partial\mu^2} = \frac{2n}{\mu^2} + \frac{n\sigma^2}{(\mu\sigma+\rho)^2}$$

$$\frac{\partial^2 lnL}{\partial\sigma\partial\mu} = \frac{-n\rho}{(\mu\sigma+\rho)^2} = \frac{\partial^2 lnL}{\partial\sigma\partial\mu}$$

$$\frac{\partial^2 lnL}{\partial\mu\partial\sigma} = \frac{n\sigma}{(\mu\sigma+\rho)^2} = \frac{\partial^2 lnL}{\partial\rho\partial\mu}$$

$$\frac{\partial^2 lnL}{\partial\sigma^2} = \frac{-n\mu^2}{(\mu\sigma+\rho)^2} - \sum_{i=1}^n \frac{1}{(\sigma+\mu z_i^2)^2}$$

$$\frac{\partial^2 \ln L}{\partial \rho^2} = \frac{n}{(\mu\sigma + \rho)^2} - \sum_{i=1}^{n} \frac{z_i^2}{(\sigma + \rho z_i)^2}$$

$$\frac{\partial^2 lnL}{\partial\sigma\partial\rho} = \frac{n\mu}{(\mu\sigma + \rho)^2} - \sum_{i=1}^{n} \frac{z_i}{(\sigma + \rho z_i)^2} = \frac{\partial^2 lnL}{\partial\rho\partial\sigma}$$

The following equation for $\hat{\mu},\ \hat{\sigma}\ and\ \hat{\rho}$ can be solved numerical

$$\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \mu^2} & \frac{\partial^2 \ln L}{\partial\mu\partial\sigma} & \frac{\partial^2 \ln L}{\partial\mu\partial\rho} \\ \frac{\partial^2 \ln L}{\partial\sigma\partial\mu} & \frac{\partial^2 \ln L}{\partial\sigma^2} & \frac{\partial^2 \ln L}{\partial\sigma\partial\rho} \\ \frac{\partial^2 \ln L}{\partial\rho\partial\mu} & \frac{\partial^2 \ln L}{\partial\rho\partial\sigma} & \frac{\partial^2 \ln L}{\partial\rho^2} \end{bmatrix}_{\substack{\hat{\mu}=\mu_0\\ \hat{\sigma}=\sigma_0\\ \hat{\rho}=\rho_0}} \begin{bmatrix} \hat{\mu}-\mu_0 \\ \hat{\sigma}-\sigma_0 \\ \hat{\rho}-\rho_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \ln L}{\partial\mu} \\ \frac{\partial \ln L}{\partial\sigma} \\ \frac{\partial \ln L}{\partial\rho} \end{bmatrix}_{\substack{\hat{\mu}=\mu_0\\ \hat{\sigma}=\sigma_0\\ \hat{\rho}=\rho_0}}$$

where μ_0, σ_0 and ρ_0 are the initial values of μ, σ and ρ respectively. These equations are solved iteratively till sufficiently close values of $\hat{\mu}, \hat{\sigma}$ and $\hat{\rho}$ are obtained**.**

4.2.Anderson – Darling

This method was proposed in (1952) by (Anderson – Darling), The Anderson – Darling estimator is another type of minimum distance estimator. The ADEs of the LD parameters are obtained by minimizing $\mathrm{A}(\sigma,\mu,\rho)$ as following:

$$\mathrm{A}(\sigma,\mu,\rho) = -n - \tfrac{1}{n}\textstyle\sum_{i=1}^{n}(2i-1)[\mathrm{Log}(z_i) + \mathrm{Log}(z_i)] \qquad (13)$$

The expression can be found by solving the following non-linear equation

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta \mathrm{s}_{(z_i)}}{\mathrm{F}(z_i)} - \frac{\Delta i(z_{n+1-i})}{s(z_{n+1-i})}\right] = 0 \qquad (14)$$

S=1,2,3

$$\Delta_1(z_i,\sigma,\mu,\rho) = \frac{\partial}{\partial\sigma}\mathrm{F}(z_i,\sigma,\mu,\rho)$$

$$\Delta_2(z_i,\sigma,\mu,\rho) = \frac{\partial}{\partial\mu}\mathrm{F}(z_i,\sigma,\mu,\rho) \qquad (15)$$

$$\Delta_3(z_i,\sigma,\mu,\rho) = \frac{\partial}{\partial\rho}\mathrm{F}(z_i,\sigma,\mu,\rho)$$

4.3.Right-Tail Anderson-Darling

we get it by minimizing $\mathrm{R}(\sigma,\mu,\rho)$as following:

$$\mathrm{R}(\sigma,\mu,\rho) = \tfrac{n}{2} - 2\textstyle\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{f}(z_{\mathrm{i:n}}\ \sigma,\mu,\rho) - \tfrac{1}{\mathrm{n}}\textstyle\sum_{\mathrm{i=1}}^{\mathrm{n}}(2\mathrm{i}-1)\mathrm{Log}\,(z_{n+1-i:n\ \sigma,\mu,\rho}) \qquad (16)$$

With respect to$\sigma, \mu\ and\ \rho$.The RADEs can also be obtained by solving the non-linear equations

$$-2\textstyle\sum_{i=1}^{n}\Delta_s\,(z_i\colon\sigma,\mu,\rho) + \tfrac{1}{\mathrm{n}}\textstyle\sum_{\mathrm{i=1}}^{\mathrm{n}}(2\mathrm{i}-1)\,\frac{\Delta_s(z_{n+1-i:n\ \sigma,\mu,\rho})}{s(z_{n+1-i:n\ \sigma,\mu,\rho})} = 0 \qquad (17)$$

S=1,2,3

where $\Delta_1(\sigma,\mu,\rho)$, $\Delta_2(\sigma,\mu,\rho)$, $\Delta_3(\sigma,\mu,\rho)$defined in the equation (15)

4.4.Maximum Product of spacing

The method of maximizing the multiplication of distances [2,8] is an approximation to an information scale Kullback – Leibler and it is a suitable alternative to the method of the maximum likelihood, let it be:

$\mathrm{D}_1(\sigma,\mu,\rho) = F\big(z_{(i)},\sigma,\mu,\rho\big) - F\big(z_{(i-1)},\sigma,\mu,\rho\big) = 1$ Where LD

And also$\sum_{i=1}^{n} \Delta_1(\sigma,\mu,\rho) = 1$ and that the maximization of distances for $(\hat{\mu}, \hat{\sigma}, \hat{\rho})$ can be obtained by maximizing the geometric mean of the distances $G(\sigma,\mu,\rho) = [\prod_{i=1}^{n+1} D_i(\sigma,\mu,\rho)]^{\frac{1}{n+1}}$

With respect to (σ,μ,ρ) or equivalently by maximizing the logarithm of the geometric mean of the sample distance

$$\mathrm{H}(\sigma,\mu,\rho) = \frac{1}{n+1}\sum_{i=1}^{n+1} \mathrm{Log} D_i(\sigma,\mu,\rho)$$

We can get MPSE for Lindley parameters by solving non-linear equations defined as follows:

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\sigma,\mu,\rho)}\left[\Delta_s\left(z_{(i)},\sigma,\mu,\rho\right) - \Delta_s\left(z_{(i-1)},\sigma,\mu,\rho\right)\right] = 0,\quad s\ 1,2,3 \qquad (18)$$

Where $\Delta_1(.|\sigma,\mu,\rho)$, $\Delta_2(.|\sigma,\mu,\rho)$, $\Delta_3(.|\sigma,\mu,\rho)$ learning in the equation (15)

4.5.The Cramer-Von mess

The method (CVME) is a type of minimum distance estimator, which has less discriminant than the other minimum [7] is obtained (CVME) on the basis of the difference between the estimates of CDF and the empirical distribution function[5] for parameters (CVME) are get values$(\sigma,\mu,\rho)$to distribute LD by reducing

$$C(\sigma,\mu,\rho) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F\left(z_{(i)}|\sigma,\mu,\rho\right) - \frac{2i-1}{2n}\right]^2 \qquad (19)$$

Where you get values(σ,μ,ρ) by solving non-linear equations

$$\sum_{i=1}^{n}\left[F\left(z_{(i)}|\sigma,\mu,\rho\right) - \frac{2i-1}{2n}\right]\Delta_s\left(z_{(i)}|\sigma,\mu,\rho\right) = 0\ ,\ \mathrm{S}=1,2,3 \qquad (20)$$

Where $\Delta_1(.|\sigma,\mu,\rho)$, $\Delta_2(.|\sigma,\mu,\rho)$, $\Delta_3(.|\sigma,\mu,\rho)$ learning in the equation (15)

4.6.ordinary Least square (OLS)

Let $z_1, z_2, \ldots, z_n$be the order statistics of sample size n from; $F(z;\sigma,\mu,\rho)$ *in* (4). *take the olse* [14].$\hat{\sigma}, \hat{\mu}$ *and* $\hat{\rho}$ can be obtained by minimizing $v(\mu,\sigma,\rho)$ as following:

$$\mathrm{V}(\mu,\sigma,\rho) = \sum_{i=1}^{n}\left[z_{(i)}\middle|\sigma,\mu,\rho\right) - \frac{i}{n+1}]^2 \qquad (21)$$

With respect to σ, μ *and* ρ or equivalently solving the following non-linear equation

$$\sum_{i=1}^{n}\left[\mathrm{F}\left(z_{(i)}\middle|\sigma,\mu,\rho\right) - \frac{i}{n+1}\right]\ \Delta S\left(z_{(i)}\middle|\mu,\sigma,\rho\right) = 0 \quad ,s = 1,2,3 \qquad (22)$$

Where

$\Delta_1(.|\sigma,\mu,\rho), \Delta_2(.|\sigma,\mu,\rho)$*and* $\Delta_3(.|\sigma,\mu,\rho)$ are defined in (15)

4.7.Weighted least square (WLS)

The WLSE (Swain et al.,1988),$\hat{\sigma}, \hat{\mu}$ *and* $\hat{\rho}$ can be obtained by minimizing the following equation

$$\mathrm{W}(\sigma,\mu,\rho) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\mathrm{F}\left(z_{(i)}\middle|\sigma,\mu,\rho\right) - \frac{i}{n+1}\right]^2 \qquad (23)$$

With respect to σ, μ *and* ρ or the WLSE can also be obtained by solving the following non-linear equation

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\mathrm{F}\left(z_{(i)}\middle|\sigma,\mu,\rho - \frac{i}{n+1}\right]\ \Delta S\left(z_{(i)}\middle|\sigma,\mu,\rho\right) = 0\ , s = 1,2,3 \qquad (24)$$

Where $\Delta_1(.|\sigma,\mu,\rho), \Delta_2(.|\sigma,\mu,\rho)$ and $\Delta_3(.|\sigma,\mu,\rho)$ are proided in (15)

4.7.percentile

Kao (1958,1959) proposed the PCE. let ui = $i/(n+1)$be an unbiased estimator of $\mathrm{F}\left(z_{(i)}\middle|\sigma,\mu,\rho\right)$. Then the PCE of the parameters of LD distribution are obtained by minimizing the following function

$$\mathrm{P}(\sigma,\mu,\rho) = \sum_{i=1}^{n}\left(z_{(i)} - \left[\frac{-\rho w\left(\frac{(u-1)(\mu\sigma+\rho)}{\rho e^{\frac{\mu z+\rho}{\rho}}}\right)+\mu\sigma+\rho}{\mu\rho}\right)^2 \qquad (25)$$

With respect to σ, μ *and* ρ**,**where w(.) is the Lambert function.

5.Simulation study

We use simulation to compare the eight estimating methods: WLS, OLS, ML, MPS, CVM, AD, RAD, and PC, by using some statistical criteria, which is; average of absolute value of biases(BIAS), average of mean square errors (MSE), and average of mean relative errors (MRE) we generated n=5000 random samples $z_1, z_2, \ldots, z_n$ of three size n= 30, 80 and 200 from the LD model by using equation (7). We used R software (version 4.0.2) [10] for each sample to estimate the parameters μ, σ and ρ of the LD. These tables show the rank of each of the estimators among all the estimators in each row, the superscripts are the indicators, and the ∑ Rank is the partial sum of the ranks for each column in s certain sample size. Table 6 shows the results of this study that is

the estimators partial and overall ranks.

Par.	n	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	PCE
	30	2	5	3	4	7	1	6	8
μ=0.2,σ=0.7,ρ=0.5	80	1	6	2.5	4	7	2.5	5	8
	200	4	5	1	3	7	2	6	8
	30	3	7	4	6	2	1	5	8
μ=0.2,σ=0.7,ρ=1	80	1	7	3	4	6	2	5	8
	200	2	5	1	3	6	4	7	8
	30	4	6	1	7	3	2	5	8
μ=0.2,σ=0.7,ρ=3	80	4	7	1	5	3	2	6	8
	200	4	6.5	1	2	5	3	6.5	8
	30	4.5	6	2	3	7	1	4.5	8
μ=0.2,σ=1.2,ρ=0.5	80	3	6	2	4	7	1	5	8
	200	4	5	1	3	7	2	6	8
	30	2	5	3	4	7	1	6	8
μ=0.2,σ=1.2,ρ=1	80	3	5.5	1.5	4	7	1.5	5.5	8
	200	4	5	1	2.5	7	2.5	6	8
∑ Ranks		45.5	87	28	58.5	88	28.5	84.5	120
Overall Rank		3	6	1	4	7	2	5	8

The results of the ranking,we show that MLE is the first,but ADE was very close to MLE. Where rank MLE=28 and rank ADE=28.5,leading the Andersons is good and close to MlE in this distribution, and we indicat that ADE outperformed MLE to estimate the parameter of stude (The Flexible Burrz –Gfmily:inference,and Application in the Engineering Science) .(Abdulhakim;A,A,Hazem Al-Mofileh , et.ca)

. demonstrating that these estimators are asymptotically unbiased the MSE and MRE are two different types of MSE. reduced as grew large indicating that these estimates are reliable.

All estimator method show consistency for all parameter combinations.

From Table 6 and the parameter combinations we conclude that the MLE outperformed all the other estimate are all secure of 28. Therefor base on our study we can consider the MLE method as the best. It is clear from the table and for the default values (μ=0.2, σ=0.7, ρ=0.5) the following results:

a. At a sample size of (30) the Anderson Darling method (ADE) was the best.

b. At a sample size of (80) the weighed least squares method (WLSE) was the best.

c. At a sample size of (200) the (MLE) is the best.

Also the same results for default values (μ=0.2, σ=0.7, ρ=1) as for default values (μ=0.2, σ=0.7, ρ=3)

The (MLE) was the best for sample sizes (30,80,200) for default values (μ=0.2, σ=1, ρ=0.5)

a. At a sample size of (30) the (ADE) is the best.

b. At a sample size of (80) the (ADE) is the best.

c. At a sample size of (200) the (MLE) is the best.

For default values (μ=0.2, σ=1, ρ=0.5)

a. At a sample size of (30) the (ADE) is the best.

b. At a sample size of (80) the MLE and ADE is the best.

c. At a sample size of (200) the (MLE) is the best.

6.Conclusion

To estimate three parameters of the Lindley distribution, Maximum Likelihood estimation method was the best according to the following criteria used (MSE, MRE, BIAS) compared with the methods used in the research (Cramer von-Mess, Anderson – Darling, Right -Tall Anderson – Darling, Maximum product spacing) Maximum Likelihood

estimation method was the best and highest, so we recommend using, but ADE was very close to MLE. where MLE=28 and ADE=28.5,which indicates Andersons forse, it is avery important method and close to MlE in this distribution.The method ADE is the strongest in the small samples for all formations parameter values and in the large samples the method MLE was the strongest and MLE was drawn on the medium sample as well as ADE was also drawn for the medium samples but MLE was the strongest.

References

1. Cheng, R. Amin. N. Maximum product of spacings Estimation with Application to the log normal Distribution Mathematical Report 79-1, university of Wales: Cardiff, UK, 1979
2. D.V Lindley "Introduction to probability and statistics from Bayesian view point " Cambridge university press, new yourk, 1965.doi :10 :1017 /cobog780511662273.
3. Gupta, R, k. and Kundu, D. (2000), Generalized exponential distribution different method of estimation, journal of statistical computation and simulation vol. OO, pp 1-22.
4. Hassan. A. (2016). A compound class of life time Distribution. British Journal of Applied Science Technology, 14 (3), 1-18.
5. Luceno. A. fitting the generalized pareto distribution data using maximum goodness. Of fit estimator compute stat. Data Anal. 2006 .51. 904-917. [Crossruff].
6. Lindley, D.V. Fiducial distributions and Bayes theorem. J. R. Stat. Soc. 1958, 20, 102–107
7. Macdonald. P. commenton "on estimation procedure for mixtures of distribution " by choi and Bulgren. J.R.stat. Soc. Ser.B Methiodal. 1971,33,326-329.
8. M. San Karan "The discrete Poisson - Lindley Distribution Biometrics, vol .26.1.1970. pp.145-149. doi: 10.230712529053.
9. Rama shanker, kamlesh shukla, Ravi shanker, Tekie Asehum Leanida "A three -parameter Lindley Distribution " Amerrca journal of Mathematica and statistics 2017, 7 (1): 15-26. doi: 10.5923 /j. ajms.20170701.03
10. R Core Team. R: A Language and Environment for Statistical Computing; R Foundation for Statistical Computing: Vienna, Austria, 2020. Available online: https://www.R-project.org/(accessed on: 22 June 2020).
11. Swain, J, Venkatraman, S and Wilson, j (1988) "least squares Estimation of distribution Function in Johnson's Translation system " Journal of statistical computation and simulation, 29,271-297.
12. Ali, A. H. (2017). Modifying Some Iterative Methods for Solving Quadratic Eigenvalue Problems. doi:10.13140/RG.2.2.13311.84645.
13. Rasheed, M., Ali, A. H., Alabdali, O., Shihab, S., Rashid, A., Rashid, T., & Hamad, S. A. (2021). The Effectiveness of the Finite Differences Method on Physical and Medical Images Based on a Heat Diffusion Equation. In Journal of Physics: Conference Series (Vol. 1999, No. 1, p. 012080). IOP Publishing. doi:10.1088/1742-6596/1999/1/012080
14. Warif B. Yahia, Ghassan E. Arif, Mohammed W. Al-Neama and Ali Hasan Ali, Traveling Salesman Problem Methods of Solution Survey, International Journal of Psychosocial Rehabilitation, Vol. 24, 2020, 8565-8581
15. Yahia, Warif B., Mohammed W. Al-Neama, and G. E. Arif. "PNACO: parallel algorithm for neighbour joining hybridized with ant colony optimization on multi-core system." Вестник Южно-Уральского государственного университета. Серия: Математическое моделирование и программирование 13.4 (2020).

16. Yaseen, M. T., Ali, A. H., & Shanan, I. A. (2019). Weighted (k, n)-arcs of Type (n–q, n) and Maximum Size of (h, m)-arcs in PG (2, q). Communications in Mathematics and Applications, 10(3), 361-368. doi:10.26713/cma.v10i3.1275
17. Yaseen, M. T., & Ali, A. H. (2020). A new upper bound for the largest complete (k, n)-arc in PG (2, 71). Journal of Physics: Conference Series (Vol. 1664, No. 1, p. 012045). IOP Publishing. doi:10.1088/1742-6596/1664/1/012045
18. Ali, A.H. (2017). Modifying Some Iterative Methods for Solving Quadratic Eigenvalue Problems. (Master's thesis, Wright State University).
19. Hadi, S. H., & Ali, A. H. (2021). Integrable functions of fuzzy cone and ξ-fuzzy cone and their application in the fixed point theorem. Journal of Interdisciplinary Mathematics, 1-12.