L^r Inequalities of Generalized Turán-type Inequalities of Polynomials

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Abstract: If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \leq 1$, with $s-$fold zero at the origin, then for every real or complex number α with $|\alpha| \geq k$ and for each $r > 0$, Dewan et al. [Southeast Asian Bulletin of Mathematics, 34(2010), 69-77] proved that

$$\left\{\int_0^{2\pi} |D_\alpha p(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}} \geq (|\alpha| - k)\left(\frac{n+sk}{1+k}\right)\left\{\int_0^{2\pi} |p(e^{i\theta})|^r \, d\theta\right\}^{\frac{1}{r}}.$$

We obtain an improvement and a generalization of the above inequality by involving certain co-efficients of $p(z)$.

Keywords: Polynomial, $s-$fold zero, Polar Derivative,L^r inequality.

1. Introduction and Statement of Results

Let $p(z)$ be a polynomial of degree n. Then, according to a famous well-known classical result due to Bernstein [5],

$$\max_{|z|=1}|p'(z)| \leq n \max_{|z|=1}|p(z)|. \tag{1.1}$$

Inequality (1.1) is sharp and equality holds if $p(z)$ has all its zeros at the origin.
Let $p(z)$ be a polynomial of degree n having all its zerosin $|z| \leq 1$, then Turán [11] proved that

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{2} \max_{|z|=1}|p(z)|. \tag{1.2}$$

The result is sharp and equality holds in (1.2) for polynomials having all their zeros on the unitcircle.
By involvingmin$_{|z|=1}|p(z)|$in inequality (1.2), Aziz and Dawood [2] improved (1.2) under the same hy-potheses of $p(z)$ that

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{2}\left[\max_{|z|=1}|p(z)| + \min_{|z|=1}|p(z)|\right]. \tag{1.3}$$

Equality holds in (1.3) for the polynomial $p(z) = \alpha z^n + \beta$, where$|\alpha| = |\beta|$.
Malik [7] generalized inequality (1.2) by considering polynomials having all theirzerosin $|z| \leq k$, $k \leq 1$,that

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{1+k} \max_{|z|=1}|p(z)|. \tag{1.4}$$

The result is best possible and the extremal polynomial is $p(z) = (z+k)^n$.
Let αbe a complex number, then the polar derivative of a polynomial $p(z)$ of degree n with respect to α, denoted as $D_\alpha p(z)$ is

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

$D_\alpha p(z)$is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in thesense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha p(z)}{\alpha} = p'(z).$$

The L^r- norm extension of inequality (1.1) for $r > 0$ is

$$\left\{\int_0^{2\pi}\left|p'\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq n\left\{\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{1.5}$$

Zygmund [12] proved inequality (1.5) for $r \geq 1$ for all trigonometric polynomials of degree n andnot only for those which are of the form $p\left(e^{i\theta}\right)$. The validity of (1.5) for $0 < r < 1$ was proved byArestov [1].
From a well-known fact of analysis [9, 10], we know that

$$\lim_{r\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} = \max_{|z|=1}|p(z)|. \tag{1.6}$$

In view of (1.6), inequality (1.5) is the L^r analogue of Bernstein's inequality (1.1). This important fact shows that L^r inequalities of a polynomial generalize ordinary inequalities of polynomials.
Aziz and Shah [3] considered the class of polynomials having all theirzerosin $|z| \leq k,\ k \leq 1$, with $s-$fold zero at the origin and proved the following result.

Theorem A. *Let $p(z)$ be a polynomial of degree n having all its zeros in $|z| \leq k, k \leq 1$, with $s-$fold zero at the origin, then*

$$\max_{|z|=1}|p'(z)| \geq \left(\frac{n+sk}{1+k}\right)\max_{|z|=1}|p(z)|. \tag{1.7}$$

The result is sharp and the extremal polynomial is $p(z) = z^s(z+k)^{n-s}, \quad 0 \leq s \leq n.$

Theorem A was generalized by Dewan et al. [6] by proving the following result in L^r inequality concerning polar derivative of the polynomial.

Theorem B. *If $p(z)$ be a polynomial of degree n having all its zeros in $|z| \leq k, k \leq 1$, with $s-$fold zero at the origin, then for every real or complex number α with $|\alpha| \geq k$, and for each $r > 0$*

$$\left\{\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \geq (|\alpha|-k)\left(\frac{n+sk}{1+k}\right)\left\{\int_0^{2\pi}|p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}. \tag{1.8}$$

In this paper we prove the following L^rversion for the polar derivative of a polynomial with $s-$fold zero at the origin which improves and generalizes Theorem B due to Dewan et al. [6].

Theorem. *Let $p(z) = z^s\sum_{\nu=0}^{n-s}a_\nu z^\nu$ be a polynomial of degree n having all its zeros in $|z| \leq k, k \leq 1$, with $s-$fold zero at the origin. Then for every real or complex number α with $|\alpha| \geq A_s$ and β with $|\beta| < k^{n-s}$, and for each $r > 0$*

$$\left\{\int_0^{2\pi}\left|D_\alpha\left\{p\left(e^{i\theta}\right)+\frac{m}{k^n}\beta e^{is\theta}\right\}\right|^r d\theta\right\}^{\frac{1}{r}} \geq (|\alpha|-A_s)\left(\frac{n+sk}{1+k}\right) \times \left\{\int_0^{2\pi}\left|p\left(e^{i\theta}\right)+\frac{m}{k^n}\beta e^{is\theta}\right|^r d\theta\right\}^{\frac{1}{r}}, \tag{1.9}$$

where

$$A_s = \frac{(n-s)|a_{n-s}|k^2+|a_{n-s-1}|}{(n-s)|a_{n-s}|+|a_{n-s-1}|} \tag{1.10}$$

and $m = \min_{|z|=k}|p(z)|.$

$$\textbf{Remark 1.}\ In\ the\ context\ of\ the\ Theorem, by\ Rouche's\ Theorem, the\ polynomial\ P(z) = p(z) + \frac{m}{k^n}\beta z^s\ has\ all\ its\ zeros\ in\ |z| \le k, k \le 1, whereas\ Q(z) = z^n \overline{P\left(\frac{1}{\bar z}\right)}\ has\ no\ zero\ in\ |z| < \frac{1}{k}, \frac{1}{k} \ge 1.$$

$Further,$

$$Q(z) = \overline{a_{n-s}} + \overline{a_{n-s-1}}z + \cdots + \overline{\left(a_0 + \frac{m}{k^n}\beta\right)} z^{n-s}$$

$is\ a\ polynomial\ of\ degree\ n-s.\ Thus, on\ applying\ Lemma$ 2.1 $to\ Q(z), it\ follows\ from$ (2.2) $for \mu = 1$ $that$

$$\frac{1}{n-s}\frac{|a_{n-s-1}|}{|a_{n-s}|}\frac{1}{k} \le 1. \tag{1.11}$$

Taking $\beta = 0$ in the Theorem, we have the following result which improves Theorem B due toDewan et al. [6].

$\textbf{Corollary 1}.$ $Let\ p(z) be\ a\ polynomial\ of\ degree\ n\ having\ all\ its\ zeros\ in\ |z| \le k, k \le 1, with\ s-fold\ zero\ at\ the\ origin, then\ for\ every\ real\ or\ complex\ number\ \alpha\ with\ |\alpha| \ge A_s,$ $and\ for\ each\ r > 0$

$$\left\{\int_0^{2\pi} |D_\alpha p(e^{i\theta})|^r\, d\theta\right\}^{\frac{1}{r}} \ge (|\alpha| - A_s)\left(\frac{n+sk}{1+k}\right)\left\{\int_0^{2\pi} |p(e^{i\theta})|^r\, d\theta\right\}^{\frac{1}{r}}, \tag{1.12}$$

$where\ A_s\ is\ as\ defined\ in\ the\ Theorem.$

$\textbf{Remark 2}.$ $To\ show\ that\ Corollary$ 1 $is\ an\ improvement\ of\ Theorem\ B, it\ is\ required\ to\ show\ that (|\alpha| - A_s) \ge (|\alpha| - k)\ for\ which\ it\ is\ sufficient\ to\ show\ that\ A_s \le k.$
$Suppose\ A_s \le k, then$

$$\frac{(n-s)|a_{n-s}|k^2 + |a_{n-s-1}|}{(n-s)|a_{n-s}| + |a_{n-s-1}|} \le k$$

$$i.e \qquad |a_{n-s-1}|(1-k) \le (n-s)k|a_{n-s}|(1-k).$$

$Since\ k \le 1, the\ above\ inequality\ reduces\ to$

$$|a_{n-s-1}| \le (n-s)k|a_{n-s}|,$$

$which\ is\ true\ due\ to$ (1.11) .

Further, letting $r \to \infty$ in inequality (1.12) of $Corollary$ 1, we have the following polar derivative analogue of Theorem A.

$\textbf{Corollary 2}.$ $Let\ p(z)\ be\ a\ polynomial\ of\ degree\ n\ having\ all\ its\ zeros\ in\ |z| \le k, k \le 1, with\ s-fold$ $zero\ at\ the\ origin, then\ for\ every\ real\ or\ complex\ number\ \alpha\ with\ |\alpha| \ge A_s$

$$\max_{|z|=1}|D_\alpha p(z)| \ge (|\alpha| - A_s)\left(\frac{n+sk}{1+k}\right)\max_{|z|=1}|p(z)|, \tag{1.13}$$

$where\ A_s\ is\ as\ defined\ in\ the\ Theorem.$

$\textbf{Remark 3}.$ $Taking\ limit\ as\ r \to \infty\ in$ (1.9)$and\ dividing\ the\ inequality\ so\ obtained\ by\ |\alpha|\ and\ consid-ering\ limit\ as\ |\alpha| \to \infty\ and\ |\beta| \to k^{n-s}, we\ have$

$$\max_{|z|=1}|p'(z)| \ge \left(\frac{n+sk}{1+k}\right)\max_{|z|=1}|p(z)| + \frac{n-s}{k^s(1+k)}\min_{|z|=1}|p(z)|,$$

which is an inequality proved by Aziz and Zargar [4, *Theorem* 1.2].

2. Lemma

We need the following lemmas to prove our results.
The first lemma is due to Qazi [8].

***Lemma* 2.1.** *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *is a polynomial of degree n having no zeros in* $|z| < k, k \ge 1$, *then*

$$|q'(z)| \ge k^{\mu+1} \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu-1} + 1}{1 + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}} |p'(z)| \quad on \quad |z| = 1 \tag{2.1}$$

and

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^\mu \le 1, \tag{2.2}$$

where

$$q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}.$$

***Lemma* 2.2.** *Let* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, *is a polynomial of degree n having all its zeros in* $|z| \le k, k \le 1$, *then for* $|z| = 1$

$$|p'(z)| \ge \frac{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}} |q'(z)|, \tag{2.3}$$

where

$$q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}.$$

Proof of Lemma 2.2. Since $p(z)$has all its zeros in $|z| \le k$, $k \le 1$, then $q(z)$ has no zero in$|z| < \frac{1}{k}$, $\frac{1}{k} \ge$ 1.Hence, applying Lemma 2.1 to the polynomial $q(z)$, we have by inequality (2.1)

$$|p'(z)| \ge \left(\frac{1}{k}\right)^{\mu+1} \frac{\frac{\mu}{n}\frac{|a_{n-\mu}|}{|a_n|}\left(\frac{1}{k}\right)^{\mu-1} + 1}{1 + \frac{\mu}{n}\frac{|a_{n-\mu}|}{|a_n|}\left(\frac{1}{k}\right)^{\mu+1}} |q'(z)| \quad on \quad |z| = 1,$$

which simplifies to

$$|p'(z)| \ge \frac{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}} |q'(z)|.$$

3. Proof of the Theorem

Proof. Since $p(z)$has all its zeros in $|z| \le k, k \le 1$, with s −fold zero at the origin, we take

$$p(z) = z^s h(z),$$

where

$$h(z) = a_0 + a_1 z + \cdots + a_{n-s} z^{n-s}$$

with$h(0) \ne 0$.Then, $h(z)$is a polynomial of degree $n - s$ having all its zeros in $|z| \le k$, $k \le 1$.
Let $P(z) = p(z) + \frac{m}{k^n}\beta z^s = z^s\left\{h(z) + \frac{m}{k^n}\beta\right\}$,where β is any real or complex number such that $|\beta| < k^{n-s}$, then by Rouche's Theorem, $P(z)$ has all its zeros in $|z| \le k$, $k \le 1$.
We claim that $P(z)$ has s −fold zero at $z = 0$, for which it is required to verify that at $z = 0$,

$$h(z) + \frac{m}{k^n}\beta \ne 0.$$

If $m = 0$, then the claim follows trivially, because $h(0) \neq 0$. Thus we assume$m \neq 0$. Now

$$\begin{aligned} m &= \min_{|z|=k} |p(z)| \\ &= \min_{|z|=k} |z^s h(z)| \\ &= k^s \min_{|z|=k} |h(z)|. \end{aligned}$$

Therefore

$$\frac{m}{k^s} = \min_{|z|=k} |h(z)| \leq |h(z)| \qquad for \quad |z| = k. \tag{3.1}$$

Since $|\beta| < k^{n-s}$ and with the help of (3.1), we have

$$\left|\beta \frac{m}{k^n}\right| = |\beta| \frac{m}{k^n} < k^{n-s} \frac{m}{k^n} = \frac{m}{k^s} \leq |h(z)| \qquad for \ |z| = k$$

i.e.

$$\left|\beta \frac{m}{k^n}\right| < |h(z)| \qquad\qquad for \ |z| = k. \tag{3.2}$$

Thus, by Rouche's Theorem, it can be concluded that all the zeros of $h(z) + \beta \frac{m}{k^n}$lie in the punctureddisk $0 < |z| < k$, which implies that $h(0) + \beta \frac{m}{k^n} \neq 0$.
Now,

$$P'(z) = sz^{s-1} \left\{ h(z) + \frac{m}{k^n} \beta \right\} + z^s h'(z),$$

which implies

$$\frac{zP'(z)}{P(z)} = s + \frac{z \frac{d}{dz} \left\{ h(z) + \frac{m}{k^n} \beta \right\}}{h(z) + \frac{m}{k^n} \beta}.$$

If $z_1, \ z_2, \ . \ . \ . \ , \ z_{n-s}$are the zeros of $h(z) + \frac{m}{k^n} \beta = H(z)$ (say), then none of them is zero and$|z_j| \leq k, k \leq 1$,for all $j = 1, \ 2, \ . \ . \ . \ , \ n - s$ and we have for each $\theta, \ 0 \leq \theta \leq 2\pi$,

$$\begin{aligned} Re \left\{ \frac{e^{i\theta} P'(e^{i\theta})}{P(e^{i\theta})} \right\} &= s + Re \left\{ \frac{e^{i\theta} H'(e^{i\theta})}{H(e^{i\theta})} \right\} \\ &= s + \sum_{j=1}^{n-s} Re \left\{ \frac{e^{i\theta}}{e^{i\theta} - z_j} \right\} \\ &\geq s + \sum_{j=1}^{n-s} \frac{1}{1 + |z_j|} \\ &\geq s + \frac{n-s}{1+k} \\ &= \frac{n+sk}{1+k}, \end{aligned}$$

for points $e^{i\theta}, \ 0 \leq \theta < 2\pi$, other than the zeros of $P(z)$. Hence, we have

$$\left|P'(e^{i\theta})\right| \geq \frac{n+sk}{1+k} \left|P(e^{i\theta})\right|, \tag{3.3}$$

for points $e^{i\theta}, \ 0 \leq \theta < 2\pi$, other than the zeros of $P(z)$. Since inequality (3.3) follows trivially for those points $e^{i\theta}, \ 0 \leq \theta < 2\pi$, which are the zeros of P(z) as well, it follows that for $|z| = 1$,

$$|P'(z)| \geq \frac{n+sk}{1+k}|P(z)|. \tag{3.4}$$

Since $P(z)$ has all its zeros in $|z| \leq k, k \leq 1$, therefore, by Lemma 2.2 for $\mu = 1$, we have

$$A_s|P'(z)| \geq |Q'(z)| \quad for \ |z| = 1, \tag{3.5}$$

where$A_s$is given by (1.10) and $Q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.
Also, for every real or complex number α, we have for$|z| = 1$,

$$\begin{aligned}|D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\ &\geq \big||\alpha||P'(z)| - |nP(z) - zP'(z)|\big| \\ &= \big||\alpha||P'(z)| - |Q'(z)|\big| (\because \ |Q'(z)| = |nP(z) - zP'(z)| \ \ for \ |z| = 1)\end{aligned} \tag{3.6}$$

Using (3.5), we have for $|z| = 1$

$$\begin{aligned}|\alpha||P'(z)| - |Q'(z)| &\geq |\alpha||P'(z)| - A_s|P'(z)| \\ &= (|\alpha| - A_s)|P'(z)|.\end{aligned} \tag{3.7}$$

(3.6) on using (3.7), we have

$$|D_\alpha P(z)| \geq (|\alpha| - A_s)|P'(z)| \ \ for \ |z| = 1. \tag{3.8}$$

Combining (3.4) and (3.8), we obtain

$$|D_\alpha P(z)| \geq (|\alpha| - A_s)\left(\frac{n+sk}{1+k}\right)|P(z)| \quad for \ |z| = 1$$

i.e.

$$\left|D_\alpha\left\{P(z) + \frac{m}{k^n}\beta z^s\right\}\right| \geq (|\alpha| - A_s)\left(\frac{n+sk}{1+k}\right)\left|P(z) + \frac{m}{k^n}\beta z^s\right| \quad for \ |z| = 1\,. \tag{3.9}$$

For each $r > 0$, and for each $\theta, \ 0 \leq \theta < 2\pi$, (3.9) equivalently gives

$$\left|D_\alpha\left\{P\left(e^{i\theta}\right) + \frac{m}{k^n}\beta e^{is\theta}\right\}\right|^r \geq (|\alpha| - A_s)^r\left(\frac{n+sk}{1+k}\right)^r\left|P\left(e^{i\theta}\right) + \frac{m}{k^n}\beta e^{is\theta}\right|^r. \tag{3.10}$$

Integrating both sides of (3.10) with respect to θ from 0 to 2π, we obtain

$$\int_0^{2\pi}\left|D_\alpha\left\{P\left(e^{i\theta}\right) + \frac{m}{k^n}\beta e^{is\theta}\right\}\right|^r d\theta \geq (|\alpha| - A_s)^r\left(\frac{n+sk}{1+k}\right)^r\int_0^{2\pi}\left|P\left(e^{i\theta}\right) + \frac{m}{k^n}\beta e^{is\theta}\right|^r d\theta,$$

from which the desired conclusion of the theorem follows.

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