Some aspects of meromorphic functions whose certain differential polynomials share a set.

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Abstract: This article deals with the most important and interesting topic of Nevanlinna theory that is subject on the unicity problem of the power of a mero-morphicfunctionswhosedifferentialpolynomialsshareasetwithitsderivative.Theresultswhich wegotinthissurveyistheimprovedandgeneralizedresultsofMengandLi[10].

Keywords: Meromorphic function; Differential polynomial; Weighted set sharing; Uniqueness.

1. Introduction

Inthisliteraturesurvey,thetermusedhereisacceptedvaluedistributiontheoryofmero-morphicfunctions,equallingthemilestoneofcomplexanalysisduringpastdecades.Itisknownthatthe readeriswell-knownaboutelementalnotesoftheNev̂anlinnatheory,suchas$T(\tau;\varphi), m(\tau;\varphi), N(\tau;\varphi)$, $S(\tau;\varphi)$etc...

Thisnotationcanbefoundin[4,12] for anytwofunctionsofmero-morphicwhichis not constant, φandψ,ƀ∈ $S(\varphi)\cap S(\psi)$,we say that $\varphi$and ψsharing the value ƀ CM(multiplicities counted), and in case we do not include the multiplicities, then $\varphi$and $\psi$are saidtosharingthevalueƀIM(multiplicitiesignored).

Weindicateby$S(\tau,\varphi)$anyfunctionsatisfying

$$S(\tau,\varphi)=o\{T(r,\varphi)\} as\ r\to\infty$$

probablybeyondwithafixeddefinitemeasure.

Definition 1.1.[8,9] *For a complexnumber* ƀ∈ $\mathbb{C}\cup\{\infty\}$, *we indicate* by E_{Ω}(ƀ,φ) *the fixedof all –points of* φ *where –point with multiplicity* $\mathfrak{m}$ *is counted* $\mathfrak{m}$ *times if* $\mathfrak{m}\le\Omega$ *and* $\Omega+1$ *times if* $\mathfrak{m}>\Omega$. *For a complex number* ƀ∈ $\mathbb{C}\cup\{\infty\}$, *thus* E_{Ω}(ƀ,φ)= E_{Ω}(ƀ,ψ), *then we say that* φ *and* ψ *sharing the value with weight* Ω. *The definition infers that if* φ,ψ *sharing a value* b *and weight* Ω, *then* z_0 *is a zero of* $\varphi-b$ *with multiplicity* $\mathfrak{m}(\le\Omega)$ *subject to a zero of* $\psi-b$ *with multiplicity* $n(>\Omega)$, *where* $\mathfrak{m}$*is not really equivalent to* n. *We write*$\varphi.\psi$*sharing* (b,Ω) *then* φ,ψ *sharing* (b,p) *for all integers* $p, 0\le p\le\Omega$. *And we note that* φ,ψ *sharing* a *and* b *IM or CM if and only if* φ,ψ *sharing* $(b,0)$ *or* (b,∞) *respectively.*

Definition 1.2. [8] Let I another set of definite factors of $\mathbb{C}\cup\{\infty\}$ and Ω a positive integersor ∞. Altogether $E_{\Omega}(I,\Omega)$ the element $\cup_{ƀ\in s}E_{\Omega}(b,f)$. Obviously $E_f(I)=E_f(I,\infty)$ as well $\bar{E}_f=(I,0)$.

In 2019, Meng and Li obtained the following outcomes.

Theorem 1. [10] Let $\varphi(z)$ and $\psi(z)$ be two non-constant mero-morphic functions, and let n,d,k be definite integers with $n>2k+\frac{3k+9}{d}$, $d\ge 2$ and I ={ƀ∈ $\mathbb{C}$:ƀ^{d} =1}, in case $E_{(\varphi^n)^{(k)}}(I,1)=E_{(\psi^n)^{(k)}}(I,1)$, subsequently classified among these retains:

1.$\varphi=c_1e^{cz}$and$\psi=c_2e^{-cz}$ for constants c_1,c_2,c other than zero, in order to enable

$(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd}=1$.

2.$\varphi=t\psi$with$t^{nd}=1$, t ∈ C.

Theorem 2. [10] Let $\varphi(z)$ and $\psi(z)$ be two non-constant mero-morphic functions, and let n,d,k be definite integers with $n > 2k + \frac{8k+14}{d}$, $d \geq 2$ and I ={ƀ ∈ ℂ : ƀ^d =1}, if $E_{(\varphi^n)^{(k)}}(I,0)= E_{(\psi^n)^{(k)}}(I,0)$, subsequently classified among these retains: (1) $\varphi = c_1 e^{cz}$ and $\psi = c_2 e^{-cz}$ for constants c_1, c_2, c other than zero, in order to enable $\varphi = c_1 e^{cz}$ and $\psi = c_2 e^{-cz}$ for constants c_1, c_2, c other than zero, in order to enable $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$. (2) $\varphi = t\psi$with$t^{nd} = 1, t \in$ C.

Fromthisperspective,weaccomplishedthesucceedingconclusion:

Theorem 3. Let $\varphi(z)$ and $\psi(z)$ be two non-constant mero-morphic functions, and let n, d, k be definite integers with$n > 2k + \frac{8k+14}{d}$, $d \geq 2$ and $I = \{$ƀ∈ ℂ:ƀ^d =1}, and considering

$Q(z)$ =ƀ$_m z^m$+ ƀ$_{m-1} z^{m-1}$+...+ƀ$_1 z$+ƀ$_o$be non-zero polynomials where ƀ$_o$,ƀ$_1$,...ƀ$_{m-1}$,ƀ$_m$are complex constants. If $E_{(\varphi^n)^{(k)}}(I,1) = E_{(\psi^n)^{(k)}}(I,1)$, subsequently classified among these retains:

1. $\varphi(z) = \frac{c_1}{\sqrt[n]{c_0}} e^{cz}$ and $\psi(z) = \frac{c_1}{\sqrt[n]{c_0}} e^{-cz}$ constants c_1, c_2, c other than zero, in order to enable $(-1)^k (c_1c_2)^n (nc)^{2k} = 1$ or $\varphi \equiv h\psi$ for a stable h ensure that $h^d = 1$.

2. φandψ persuade the mathematical equation $R(\varphi, \psi) \equiv 0$, whereby.

$R(\varpi_1, \varpi_2) = \varpi_1^n$(ƀ$_m \varpi_1$+ ƀ$_{m-1} \varpi_1{}^{m-1}$+...+ƀ$_1 \varpi_1$+ƀ$_o$) -$\varpi_2^n$(ƀ$_m \varpi_2$+ ƀ$_{m-1} \varpi_2{}^{m-1}$+...+ƀ$_1 \varpi_2$+ƀ$_o$).

Theorem 4. Let$\varphi(z)$ and $\psi(z)$ be two non-constant mero-morphic functions, and let n, d, k be definite integers with $n > 2k + \frac{6k+2m+8}{2}$, $d \geq 2$ and I ={ƀ∈ ℂ:ƀ^d =1}, and considering $Q(z)$ be defined in Theorem 3. If $E_{(\varphi^n Q(\varphi))^{(k)}}(I,0)=E_{(\psi^n Q(\psi))^{(k)}}(I,0)$, subsequently conviction of Theorem 3 retains.

2. Some Lemmas

Throughout this, we present few lemmas which will be essential in the development. We propose $\mathcal{R}$ as:

$$\mathcal{R} = \left(\frac{\mathcal{U}^{(\alpha)}}{\mathcal{U}^{(\beta)}} - \frac{2\mathcal{U}^{(\beta)}}{\mathcal{U}-1}\right) - \left(\frac{\mathcal{V}^{(\alpha)}}{\mathcal{V}^{(\beta)}} - \frac{2\mathcal{V}^{(\beta)}}{\mathcal{V}-1}\right). \, where \ \alpha = 1, \beta = 1$$

Lemma 1. [2]Let are $\mathcal{R}$ be defined at the top. In case $\mathcal{U}$ as well $\mathcal{V}$ sharing (1,1) plus $\mathcal{R} \not\equiv 0$.
Then,

$$T(\tau, \mathcal{U}) \leq N_2\left(\tau, \frac{1}{\mathcal{U}}\right) + N_2(\tau, \mathcal{U}) + N_2\left(\tau, \frac{1}{\mathcal{V}}\right) + N_2(\tau, \mathcal{V}) + \frac{1}{2}\overline{N}\left(\tau, \frac{1}{\mathcal{U}}\right) + \frac{1}{2}\overline{N}(\tau, \mathcal{U}) + \mathcal{S}(\tau, \mathcal{U}) + \mathcal{S}(\tau, \mathcal{V}).$$

the similar inequality remains for $T(\tau, \mathcal{V})$.

Lemma 2.[2] Let are $\mathcal{R}$ be defined at the top. In case $\mathcal{U}$ as well $\mathcal{V}$ sharing (1,0) plus $\mathcal{R} \not\equiv 0$.
Then,

$$T(\tau, \mathcal{U}) \leq N_2\left(\tau, \frac{1}{\mathcal{U}}\right) + N_2(\tau, \mathcal{U}) + N_2\left(\tau, \frac{1}{\mathcal{V}}\right) + N_2(\tau, \mathcal{V}) + 2\overline{N}(\tau, \mathcal{U}) + \overline{N}\left(\tau, \frac{1}{\mathcal{V}}\right) + \overline{N}(\tau, \mathcal{V}) + \mathcal{S}(\tau, \mathcal{U}) + \mathcal{S}(\tau, \mathcal{V}).$$

the similar inequality remains for $T(\tau, \mathcal{V})$.

Lemma 3.[2]Letφbeanon-constantmero-morphicfunctionandn, m, κbepositiveintegerswith$n + m > l$.Ifz$_0$isapoleofφ,as$p + q$isitsmultiplicityinsomeneighbourhoodofz$_0$.

$$\frac{(\varphi^n Q(\varphi))^{(k)}}{\varphi^{n+m-k}} = \bar{h}_k (z - z_0)^{(p+q-1)k},$$

where$\bar{h}_k$ is regular z_0 and $\bar{h}_k(z_0) \neq 0$, the proof is straight forward and so it is omitted.

Lemma 4. [2] Letφbeanon-constantmero-morphicfunctionandn, lbepositiveintegerswith $n + m > l$.Ifz$_0$isapoleofφ,as$p + q$isitsmultiplicityinsomeneighbourhoodofz$_0$.

$$\frac{(\varphi^n Q(\varphi))^{(k)}}{\varphi^{n+m-k}} = \bar{h}_k (z - z_0)^{(p+q-1)k},$$

where$\bar{h}_k$ is regular z_0 and $\bar{h}_k(z_0) \neq 0$, the proof is straight forward and so it is omitted.

Lemma 5. Letφbeanon-constantmero-morphicfunctionandn, m, κbepositiveintegerswith $n + m > 2k$. Then

$i.(n + m - 2k)T(\tau, \varphi) + kN(\tau, \varphi) + N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n Q(\varphi))^{(k)}}\right) \leq T\left(\tau, \left(\varphi^n Q(\varphi)\right)^{(k)}\right) + \mathcal{S}(\tau, \varphi)$

$ii. N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n Q(\varphi))^{(k)}}\right) \leq kN(\tau, \varphi) + \mathcal{S}(\tau, \varphi).$

Proof.We notice that

$$N\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)=(n+m)N(\tau.\varphi)+k\bar{N}(\tau,\varphi)$$
$$T\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)=(n+m)T(\tau,\varphi)$$

so that $S\left(\tau,\varphi^{n}Q(\varphi)\right)=\mathcal{S}(\tau,\varphi)$.
And by Milloux theorem we bring

$$m\left(\tau,\frac{\varphi^{(j)}}{\varphi}\right)=\mathcal{S}(\tau,\varphi)$$

for$j=1,2,\dots l$. Now
$(n+m-k)m(\tau,\varphi)=m(\tau,\varphi^{n+m-k})$

$$\le m\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)+m\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}\right)+\mathcal{S}(\tau,\varphi)$$
$$\le\ m\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)+T\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}\right)-N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}\right)+\ \mathcal{S}(\tau,\varphi).$$

We learn that,
$T(\tau,\varphi)=T\left(\tau,\frac{1}{\varphi}\right)+O(1)$,

$$\le m\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)+T\left(\tau,\frac{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}{\varphi^{n+m-k}}\right)-N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}\right)+O(1)$$
$$=\ m\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)+m\left(\tau,\frac{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}{\varphi^{n+m-k}}\right)+N\left(\tau,\frac{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}{\varphi^{n+m-k}}\right)-N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}\right)$$
$$+O(1)$$
$$\le\ m\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)+km(\tau,\varphi)+\ N\left(\tau,\frac{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}{\varphi^{n+m-k}}\right)-N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}\right)+\mathcal{S}(\tau,\varphi)$$

By Lemmas 3 & 4, we reach

$$(n+m-k)m(\tau,\varphi)\le\ m\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)+km(\tau,\varphi)+kN(\tau,\varphi)+k\bar{N}(\tau,\varphi)$$
$$-N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}\right)+\ \mathcal{S}(\tau,\varphi)$$
$$=\ m\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)+N\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)-(n+m)N(\tau,\varphi)+kT(\tau,\varphi)$$
$$-N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}\right)+\ \mathcal{S}(\tau,\varphi).$$

and so

$$(n+m-2k)\ T(\tau,\varphi)+k\ N(\tau,\varphi)+\ N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}\right)\le T\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)+\ \mathcal{S}(\tau,\varphi)$$

which is (i).
Again with Lemma 3 one can notice that $\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}$ is regular at each pole of φ. So poles of $\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}$ occur only at the zeros of $\left(\varphi^{n}Q(\varphi)\right)^{(k)}$. We observe by Lemma 4 that a zero of $\left(\varphi^{n}Q(\varphi)\right)^{(k)}$, which is additionally a zero of φ with multiplicity p, is a pole of $\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}$ as $(p+q-1)k$ is its multiplicity. Thus

$$N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}/\varphi=0\right)=kN(\tau,0;\varphi)-k\bar{N}(\tau,0;\varphi).$$

Also

$$N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}/\varphi\neq0\right)=N(\tau,0;\left(\varphi^{n}Q(\varphi)\right)^{(k)}|\varphi\neq0)\le\ N\left(\tau,0;\frac{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}{\varphi^{n}Q(\varphi)}\right)$$

$$\le\ T\left(\tau,\frac{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}{\varphi^{n}Q(\varphi)}\right)=\ N\left(\tau,\frac{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}{\varphi^{n}Q(\varphi)}\right)+\ \mathcal{S}(\tau,\varphi)$$

$$=k\overline{N}(\tau,0;\varphi)+k\overline{N}(\tau,\varphi)+\mathcal{S}(\tau,\varphi).$$

Therefore,

$$N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}\right)=N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}/\varphi=0\right)+N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}/\varphi\neq 0\right)$$

$$\leq k\overline{N}(\tau,0;\varphi)+k\overline{N}(\tau,\varphi)+\mathcal{S}(\tau,\varphi),$$

as (ii) follows.

Lemma 6. [13] Letφ be a non-constant mero-morphic function, k be a define integer, afterwards.

$$N_{\omega}\left(\tau,\frac{1}{\varphi^{(k)}}\right)\leq N_{\omega+k}\left(\tau,\frac{1}{\varphi^{(k)}}\right)+k\overline{N}(\tau,\varphi)+\mathcal{S}(\tau,\varphi),$$

where$N_{\omega}\left(\tau,\frac{1}{\varphi^{(k)}}\right)$ represents the function counting the zeros of $\varphi^{(k)}$ whereby a zero of multiplicity m is counted m times if $m\leq\omega$ likewise ω times if $m>\omega$. Certainly,

$$N_{\omega}\left(\tau,\frac{1}{\varphi^{(k)}}\right)=N_{1}\left(\tau,\frac{1}{\varphi^{(k)}}\right).$$

Lemma 7. [7] Let φ and ψ be two non-constant entire functions, and let n,l be two definite integers with $n>l$, also considering $Q(z)=\text{ь}_{m}z^{m}+\text{ь}_{m-1}z^{m-1}+\ldots+\text{ь}_{1}z+\text{ь}_{o}$ be a non-zero polynomials, where $\text{ь}_{o},\text{ь}_{1},\ldots\text{ь}_{m-1}$are complex constants. If $\left(\varphi^{n}Q(\varphi)\right)^{(l)}\left(\psi^{n}Q(\psi)\right)^{(l)}\equiv 1$ then $Q(z)$ reduces to a non-zero monomials, i.e., $Q(z)=\text{ь}_{i}z^{i}\not\equiv 0$ for some $i=0,1,2,,..m$. Moreover $\varphi(z)=\frac{d_{1}}{\sqrt[n+1]{a_{i}}}\exp(cz)$ and $\psi(z)=\frac{d_{1}}{\sqrt[n+1]{a_{i}}}\exp(-cz)$ where d_{1},d_{2},c are three constants to enable $(-1)^{l}(d_{1}d_{2})^{(n+i)}\{(n+i)c\}^{2l}=1$.

2. Proof of Theorems

Proof of Theorem 3. Let

$$\mathcal{U}=(\left(\varphi^{n}Q(\varphi)\right)^{(k)})^{d}\qquad \mathcal{V}=(\left(\psi^{n}Q(\psi)\right)^{(k)})^{d}. \tag{1}$$

Since, $\mathrm{E}_{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}(\mathcal{S},1)=\mathrm{E}_{\left(\psi^{n}Q(\psi)\right)^{(k)}}(\mathcal{S},1)$, we see that $\mathcal{U}$ and $\mathcal{V}$ sharing (1,1). If $\mathcal{R}\not\equiv 0$ then by Lemma 1,

$$T(\tau,\mathcal{R})\leq N_{2}\left(\tau,\frac{1}{\mathcal{U}}\right)+N_{2}(\tau,\mathcal{U})+N_{2}\left(\tau,\frac{1}{\mathcal{V}}\right)+N_{2}(\tau,\mathcal{V})+\frac{1}{2}\overline{N}\left(\tau,\frac{1}{\mathcal{U}}\right)+\mathcal{S}(\tau,\mathcal{U})+\mathcal{S}(\tau,\mathcal{V}) \tag{2}$$

By Lemma 5, we gain,

$$(n+m-2k)T(\tau,\varphi)\leq T\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)+\mathcal{S}(\tau,\varphi) \tag{3}$$
$$\leq(k+1)(n+m)\,T(\tau,\varphi)+\mathcal{S}(\tau,\varphi)$$

$$(n+m-2k)T(\tau,\psi)\leq T\left(\tau,\left(\psi^{n}Q(\psi)\right)^{(k)}\right)+\mathcal{S}(\tau,\varphi) \tag{4}$$
$$\leq(k+1)(n+m)\,T(\tau,\varphi)+\mathcal{S}(\tau,\varphi)$$

Since

$$T\left(\tau,\left(\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)^{d}\right)=dT\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)+\mathcal{S}\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right) \tag{5}$$
$$T\left(\tau,\left(\left(\psi^{n}Q(\psi)\right)^{(k)}\right)^{d}\right)=dT\left(\tau,\left(\psi^{n}Q(\psi)\right)^{(k)}\right)+\mathcal{S}\left(\tau,\left(\psi^{n}Q(\psi)\right)^{(k)}\right) \tag{6}$$

It is easy to see that

$$\mathcal{S}\left(\tau,\left(\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)^{d}\right)=\mathcal{S}\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)+\mathcal{S}(\tau,\varphi) \tag{7}$$
$$\mathcal{S}\left(\tau,\left(\left(\psi^{n}Q(\psi)\right)^{(k)}\right)^{d}\right)=\mathcal{S}\left(\tau,\left(\psi^{n}Q(\varphi)\right)^{(k)}\right)+\mathcal{S}(\tau,\psi) \tag{8}$$

On the other hand

$$N_2(\tau,\mathcal{U}) = N_2\left(\tau,\left(\left(\varphi^{n}\mathcal{Q}(\varphi)\right)^{(k)}\right)^{d}\right) = 2\overline{N}(\tau,\varphi) \quad (9)$$

$$N_2(\tau,\mathcal{V}) = N_2\left(\tau,\left(\left(\psi^{n}\mathcal{Q}(\psi)\right)^{(k)}\right)^{d}\right) = 2\overline{N}(\tau,\psi) \quad (10)$$

$$\frac{1}{2}\overline{N}(\tau,\mathcal{U}) = \frac{1}{2}\overline{N}(\tau,\varphi) \quad (11)$$

By Lemma 6, we took

$$\begin{aligned} N_2\left(\tau,\frac{1}{\mathcal{U}}\right) &= N_1\left(\tau,\frac{1}{\left(\left(\varphi^{n}\mathcal{Q}(\varphi)\right)^{(k)}\right)^{d}}\right) = 2\overline{N}\left(\tau,\frac{1}{\left(\varphi^{n}\mathcal{Q}(\varphi)\right)^{(k)}}\right) \\ &\le 2N_{k+1}\left(\tau,\frac{1}{(\varphi^{n}\mathcal{Q}(\varphi))}\right) + 2k\overline{N}\left(\tau,\varphi^{n}\mathcal{Q}(\varphi)\right) + \mathcal{S}(\tau,\varphi) \quad (12) \\ &\le 2(k+1)\overline{N}\left(\tau,\frac{1}{\varphi}\right) + mN\left(\tau,\frac{1}{\varphi}\right) + 2k\overline{N}(\tau,\varphi) + \mathcal{S}(\tau,\varphi). \end{aligned}$$

$$\begin{aligned} \frac{1}{2}\overline{N}\left(\tau,\frac{1}{\mathcal{U}}\right) &= \frac{1}{2}\overline{N}\left(\tau,\frac{1}{\left(\left(\varphi^{n}\mathcal{Q}(\varphi)\right)^{(k)}\right)^{d}}\right) = \frac{1}{2}\overline{N}\left(\tau,\frac{1}{\left(\varphi^{n}\mathcal{Q}(\varphi)\right)^{(k)}}\right) \\ &\le \frac{1}{2}N_{k+1}\left(\tau,\frac{1}{\varphi^{n}\mathcal{Q}(\varphi)}\right) + \frac{k}{2}\overline{N}\left(\tau,\frac{1}{\varphi^{n}\mathcal{Q}(\varphi)}\right) + \mathcal{S}(\tau,\varphi^{n}) \quad (13) \\ &\le \frac{k+1}{2}\overline{N}\left(\tau,\frac{1}{\varphi}\right) + mN\left(\tau,\frac{1}{\varphi}\right) + \frac{k}{2}\overline{N}(\tau,\varphi) + \mathcal{S}(\tau,\varphi). \end{aligned}$$

And

$$N_2\left(\tau,\frac{1}{\mathcal{V}}\right) = 2\overline{N}\left(\tau,\frac{1}{\left(\left(\psi^{n}\mathcal{Q}(\psi)\right)^{(k)}\right)^{d}}\right) \le 2\left(\overline{N}\left(\tau,\frac{1}{\psi^{n+m-k}}\right)\right) + N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}\mathcal{Q}(\psi)\right)^{(k)}}\right) \quad (14)$$

$$\le 2\left(\overline{N}\left(\tau,\frac{1}{\psi}\right)\right) + N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}\mathcal{Q}(\psi)\right)^{(k)}}\right)$$

$$N_2\left(\tau,\frac{1}{\mathcal{U}}\right) \le 2\left(\overline{N}\left(\tau,\frac{1}{\varphi}\right)\right) + N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}\mathcal{Q}(\psi)\right)^{(k)}}\right). \quad (15)$$

Combining (2), (9), (10), (11), (12), (13) and (14), we deduct

$$\begin{aligned} T\left(\tau,\left(\left(\varphi^{n}\mathcal{Q}(\varphi)\right)^{(k)}\right)^{d}\right) &\le \frac{5k+4m+5}{2}\overline{N}\left(\tau,\frac{1}{\varphi}\right) + \frac{5k+5}{2}\overline{N}(\tau,\varphi) + 2\overline{N}(\tau,\psi) + 2\overline{N}\left(\tau,\frac{1}{\psi}\right) \\ &+2N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}\mathcal{Q}(\psi)\right)^{(k)}}\right) + \mathcal{S}(\tau,\varphi) + \mathcal{S}(\tau,\psi) \quad (16) \\ &\le (3k+2m+3)T(\tau,\varphi) + 4T(\tau,\psi) + 2k\overline{N}(\tau,\varphi) \\ &+2N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}\mathcal{Q}(\psi)\right)^{(k)}}\right) + \mathcal{S}(\tau,\varphi) + \mathcal{S}(\tau,\psi). \end{aligned}$$

Similarly, we have

$$T\left(\tau,\left(\left(\varphi^{n}\mathcal{Q}(\varphi)\right)^{(k)}\right)^{d}\right) \le (4k+2m+3)T(\tau,\psi) + 4T(\tau,\varphi) + 2k\overline{N}(\tau,\psi)$$

$$+2N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}Q(\psi)\right)^{(k)}}\right)+\ \mathcal{S}(\tau,\varphi)+\mathcal{S}(\tau,\psi). \tag{17}$$

By Lemma 5, we took

$$(n+m-2k)d\,T(\tau,\varphi)+kdN(\tau,\varphi)+d\,N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}\right)\leq dT\left(\tau,\left(\varphi^{n}Q(\varphi)\right)^{(k)}\right)+\ \mathcal{S}(\tau,\varphi). \tag{18}$$

$$(n+m-2k)d\,T(\tau,\psi)+kdN(\tau,\psi)+d\,N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}Q(\psi)\right)^{(k)}}\right)\leq dT\left(\tau,\left(\psi^{n}Q(\psi)\right)^{(k)}\right)+\ \mathcal{S}(\tau,\psi). \tag{19}$$

From (16), (17), (18) and (19), we have

$$(n+m-2k)d\,T(\tau,\varphi)+kdN(\tau,\varphi)+d\,N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}\right)+(n+m-2k)d\,T(\tau,\psi)+kdN(\tau,\psi)+$$
$$d\,N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}Q(\psi)\right)^{(k)}}\right)\leq(4k+2m+3)T(\tau,\varphi)+4T(\tau,\psi)+2k\overline{N}(\tau,\varphi)+2N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}Q(\psi)\right)^{(k)}}\right)$$
$$+(4k+2m+3)T(\tau,\psi)+4T(\tau,\varphi)+2k\overline{N}(\tau,\psi)+2N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}Q(\psi)\right)^{(k)}}\right)$$
$$+\mathcal{S}(\tau,\varphi)+\mathcal{S}(\tau,\psi). \tag{20}$$

Since $d\geq 2$

$$dN\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}\right)\geq 2N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}Q(\varphi)\right)^{(k)}}\right) \tag{21}$$

$$dN\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}Q(\psi)\right)^{(k)}}\right)\geq 2N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}Q(\psi)\right)^{(k)}}\right) \tag{22}$$

$$kdN(\tau,\varphi)\geq 2k\overline{N}(\tau,\varphi) \tag{23}$$

$$kdN(\tau,\psi)\geq 2k\overline{N}(\tau,\psi) \tag{24}$$

Therefore

$$(nd+md-2kd-4k-2m-7)T(\tau,\varphi)+(nd+md-2kd-4k-2m-7)T(\tau,\psi)$$
$$\leq\ \mathcal{S}(\tau,\varphi)+\mathcal{S}(\tau,\psi). \tag{25}$$

Which contradicts $n+m>2k+\frac{6k+2m+7}{.d}$. Hence $\mathcal{R}\equiv 0$. By integrating, we must

$$\frac{1}{\mathcal{V}-1}=\frac{\mathcal{A}^{*}}{\mathcal{U}-1}+\mathcal{B}^{*}. \tag{26}$$

Where $\mathcal{A}^{*}\neq 0$ as well as $\mathcal{B}^{*}$ are consistence. Thus

$$\mathcal{V}=\frac{(\mathcal{B}^{*}+1)\mathcal{U}+(\mathcal{A}^{*}-\mathcal{B}^{*}-1)}{\mathcal{B}^{*}\mathcal{U}+(\mathcal{A}^{*}-\mathcal{B}^{*})} \tag{27}$$

and

$$\mathcal{U}=\frac{(\mathcal{B}^{*}-\mathcal{A}^{*})\mathcal{V}+(\mathcal{A}^{*}-\mathcal{B}^{*}-1)}{\mathcal{B}^{*}\mathcal{V}-(\mathcal{B}^{*}+1)} \tag{28}$$

further in our view can have three sub cases:

Sub case 1: $\mathcal{B}^* \neq 0, -1$. Then from (28) we may

$$\overline{N}\left(\tau, \frac{1}{\mathcal{V}-\frac{\mathcal{B}^*+1}{\mathcal{B}^*}}\right) = \overline{N}(\tau, \mathcal{U}). \tag{29}$$

By of Nevanlinna second elementary theorem and (14), can take

$$T(\tau,\mathcal{V}) \leq \overline{N}(\tau,\mathcal{V}) + \overline{N}\left(\tau,\frac{1}{\mathcal{V}}\right) + \overline{N}\left(\tau,\frac{1}{\mathcal{V}-\frac{\mathcal{B}^*+1}{\mathcal{B}^*}}\right) + \mathcal{S}(\tau,\mathcal{V})$$

$$\leq \overline{N}(\tau,\mathcal{V}) + \overline{N_2}\left(\tau,\frac{1}{\mathcal{V}}\right) + \overline{N}(\tau,\mathcal{U}) + \mathcal{S}(\tau,\mathcal{V})$$

$$\leq \overline{N}(\tau,\psi) + 2\left(\overline{N}\left(\tau,\frac{1}{\psi}\right) + N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}}\right)\right) + \overline{N}(\tau,\varphi) + \mathcal{S}(\tau,\psi). \tag{30}$$

If $\mathcal{A}^* - \mathcal{B}^* \neq 0$, as a result of (27) that

$$\overline{N}\left(\tau, \frac{1}{\mathcal{U}-\frac{\mathcal{B}^*+1-\mathcal{A}^*}{\mathcal{B}^*+1}}\right) = \overline{N}\left(\tau, \frac{1}{\mathcal{V}}\right). \tag{31}$$

Again by Nevanlinna second elementary theorem, also (15),

$$T(\tau,\mathcal{U}) \leq \overline{N}(\tau,\mathcal{U}) + \overline{N}\left(\tau,\frac{1}{\mathcal{U}}\right) + \overline{N}\left(\tau,\frac{1}{\mathcal{U}-\frac{\mathcal{B}^*+1-\mathcal{A}^*}{\mathcal{B}^*+1}}\right) + \mathcal{S}(\tau,\mathcal{U})$$

$$\leq \overline{N}(\tau,\mathcal{U}) + \overline{N_2}\left(\tau,\frac{1}{\mathcal{U}}\right) + \overline{N}(\tau,\mathcal{V}) + \mathcal{S}(\tau,\mathcal{U})$$

$$\leq \overline{N}(\tau,\varphi) + 2\left(\overline{N}\left(\tau,\frac{1}{\varphi}\right) + N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}}\right)\right)$$

$$+N_{k+1}\left(\tau,\frac{1}{\psi^{\mathrm{n}}\mathcal{Q}(\psi)}\right) + k\overline{N}(\tau,\psi) + \mathcal{S}(\tau,\psi) \tag{32}$$

$$\leq \overline{N}(\tau,\varphi) + 2\left(\overline{N}\left(\tau,\frac{1}{\varphi}\right) + N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}}\right)\right) + (k+1)N\left(\tau,\frac{1}{\psi}\right)$$

$$+m\,N\left(\tau,\frac{1}{\psi}\right) + k\overline{N}(\tau,\psi) + \mathcal{S}(\tau,\psi).$$

From (18), (19), (30) and (32), we get

$$(n+m-2k)dT(\tau,\varphi) + kdN(\tau,\varphi) + dN\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}}\right) + (n+m-2k)dT(\tau,\psi) + kdN(\tau,\psi)$$

$$+ dN\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}}\right) \leq \overline{N}(\tau,\varphi) + 2\left(\overline{N}\left(\tau,\frac{1}{\varphi}\right) + N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}}\right)\right)$$

$$+(k+1)\overline{N}\left(\tau,\frac{1}{\psi}\right) + mN\left(\tau,\frac{1}{\psi}\right) + k\overline{N}(\tau,\psi) \tag{33}$$

$$+\overline{N}(\tau,\psi)+2\left(\overline{N}\left(\tau,\frac{1}{\psi}\right)+N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}\mathcal{Q}(\psi)\right)^{(k)}}\right)\right)$$

$$+\overline{N}(\tau,\varphi)+2N\left(\tau,\frac{1}{\psi}\right)+\ \mathcal{S}(\tau,\varphi)+\mathcal{S}(\tau,\psi).$$

Since $d\geq 2$

$$dN\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}\mathcal{Q}(\varphi)\right)^{(k)}}\right)\geq 2N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}\mathcal{Q}(\varphi)\right)^{(k)}}\right). \quad (34)$$

$$dN\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}\mathcal{Q}(\psi)\right)^{(k)}}\right)\geq 2N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{n}\mathcal{Q}(\psi)\right)^{(k)}}\right). \quad (35)$$

$$kdN(\tau,\varphi)\geq 2k\overline{N}(\tau,\varphi) \quad (36)$$

$$kdN(\tau,\psi)\geq 2k\overline{N}(\tau,\psi) \quad (37)$$

Therefore

$$(nd+md-2kd-4k-2m-7)T(\tau,\varphi)+(nd+md-2kd-4k-2m-7)T(\tau,\psi)$$
$$\leq\ \mathcal{S}(\tau,\varphi)+\mathcal{S}(\tau,\psi). \quad (38)$$

Which contradicts $n+m>2k+\frac{6k+2m+7}{d}$. Moreover $\mathcal{A}^{*}-\mathcal{B}^{*}-1\equiv 0$. Then by (27)

$$\overline{N}\left(\tau,\frac{1}{\mathcal{U}+\frac{1}{\mathcal{B}^{*}}}\right)=\overline{N}(\tau,\mathcal{V}). \quad (39)$$

Again by Nevanlinna second elementary theorem, also (15)

$$T(\tau,\mathcal{U})\leq\ \overline{N}(\tau,\mathcal{U})+\overline{N}\left(\tau,\frac{1}{\mathcal{U}}\right)+\overline{N}\left(\tau,\frac{1}{\mathcal{U}+\frac{1}{\mathcal{B}^{*}}}\right)+\mathcal{S}(\tau,\mathcal{U})$$

$$\leq\overline{N}(\tau,\mathcal{U})+\overline{N_{2}}\left(\tau,\frac{1}{\mathcal{U}}\right)+\overline{N}(\tau,\mathcal{V})+\mathcal{S}(\tau,\mathcal{U})$$

$$\leq\overline{N}(\tau,\varphi)+2\left(\overline{N}\left(\tau,\frac{1}{\varphi}\right)+N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{n}\mathcal{Q}(\varphi)\right)^{(k)}}\right)\right)+\overline{N}(\tau,\psi)+\mathcal{S}(\tau,\varphi). \quad (40)$$

Combine (18), (19), (30) and (40), Therefore

$$(nd+md-2kd-2)T(\tau,\varphi)+(nd+md-2kd-2)T(\tau,\psi)\leq\ \mathcal{S}(\tau,\varphi)+\mathcal{S}(\tau,\psi). \quad (41)$$

which violates our given assumption.

Sub case 2:$\mathcal{B}^{*}=-1$. Then

$$\mathcal{V}=\frac{\mathcal{A}^{*}}{\mathcal{A}^{*}+1-\mathcal{U}} \quad (42)$$

and

$$\mathcal{U}=\frac{(1+\mathcal{A}^{*})\mathcal{V}-\mathcal{A}^{*}}{\mathcal{V}}. \quad (43)$$

If $\mathcal{A}^{*}+1\neq 0$. we obtain

$$\overline{N}\left(\tau,\frac{1}{\mathcal{U}-\mathcal{A}^*-1}\right)=\overline{N}(\tau,\mathcal{V}). \tag{44}$$

$$\overline{N}\left(\tau,\frac{1}{\mathcal{V}-\frac{\mathcal{A}^*}{\mathcal{A}^*+1}}\right)=\overline{N}\left(\tau,\frac{1}{\mathcal{U}}\right). \tag{45}$$

By similar argument we can obtain a contradiction. Therefore $\mathcal{A}^*+1=0$, then $\mathcal{U}\mathcal{V}\equiv 1$ that is

$$\left(\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}\left(\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}=1.$$

By Lemma 7, we have $\varphi(z)=\frac{d_1}{\sqrt[n+1]{a_1}}e^{cz}$ and $\psi(z)=\frac{d_1}{\sqrt[n+1]{a_1}}e^{-cz}$ where the constants d,d_2,c enables $(-1)^k(d_1d_2)^{(n+i)}\{(n+i)c^{2k}=1$.

Sub case 3: $\mathcal{B}^*=0$.Then (27) and (28) gives $\mathcal{V}=\frac{\mathcal{U}+\mathcal{A}^*-1}{\mathcal{A}^*}$and $\mathcal{U}=\mathcal{A}^*\mathcal{V}+1-\mathcal{A}^*$. If $\mathcal{A}^*-1\neq 0$, then

$$\overline{N}\left(\tau,\frac{1}{\mathcal{U}-\mathcal{A}^*-1}\right)=\overline{N}(\tau,\frac{1}{\mathcal{V}}). \tag{46}$$

$$\overline{N}\left(\tau,\frac{1}{\mathcal{V}-\frac{1-\mathcal{A}^*}{\mathcal{A}^*}}\right)=\overline{N}\left(\tau,\frac{1}{\mathcal{U}}\right). \tag{47}$$

ProceedingsimilarlyasinSubcase1,wecometoacontradiction. Therefore$\mathcal{A}^*-1=0$, then $\mathcal{U}\equiv\mathcal{V}$, that is, $\left(\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}=\left(\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}$, we have

$$\varphi^n(\mathrm{a_m}\varphi^{\mathrm{m}}+\cdots+\mathrm{a_0})=\psi^n(\mathrm{a_m}\psi^{\mathrm{m}}+\cdots+\mathrm{a_0}) \tag{48}$$

Considering $\bar{h}=\frac{\varphi}{\psi}$. If $\bar{h}$ is stable, using $\varphi=\psi\bar{h}$in (48) we perform

$$\mathrm{a_m}\psi^{\mathrm{n+m}}\left(\bar{h}^{n+m}-1\right)+\mathrm{a_{m-1}}\psi^{\mathrm{n+m-1}}\left(\bar{h}^{n+m}-1\right)+\cdots+\mathrm{a_0}\psi^{\mathrm{n}}\left(\bar{h}^{n}-1\right)=0$$

which indicates $\bar{h}^d=1$, when $d=(n+m,..n+m-i,\dots n), a_{m-i}\neq 0$for few $i=0,1,\dots m$. Thus $\varphi(z)=\psi(z)\bar{h}$ for a constant $\bar{h}$such a way that $\bar{h}^d=1$ where $d=(n+m,..n+m-i,\dots n), a_{m-i}\neq 0$ for few $i=0,1,\dots m$. If $\bar{h}$ is not a stable, then we understand by (48) that φ and ψ satisfy the algebraic equation $R(\varphi,\psi)\equiv 0$, where
$R(\varpi_1\varpi_2)={\varpi_1}^n(a_{\mathrm{m}}{\varpi_1}^{\mathrm{m}}+a_{\mathrm{m-1}}{\varpi_1}^{\mathrm{m-1}}+\cdots+a_1\varpi_1+a_0)-{\varpi_2}^n(a_{\mathrm{m}}{\varpi_2}^{\mathrm{m}}+a_{\mathrm{m-1}}{\varpi_2}^{\mathrm{m-1}}+\cdots+a_1\varpi_2+a_0)$.

Proof of Theorem 4. Considering

$$\mathcal{U}=\left(\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}\qquad\qquad \mathcal{V}=\left(\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}} \tag{49}$$

since $\mathrm{E}_{(\varphi^{\mathrm{n}})^{(\mathrm{k})}}(\mathcal{S},0)=\mathrm{E}_{(\psi^{\mathrm{n}})^{(\mathrm{k})}}(\mathcal{S},0)$, we see that $\mathcal{U}$ and $\mathcal{V}$ sharing (1,0). If $\mathcal{R}\not\equiv 0$ then by Lemma 1

$$T(\tau,\mathcal{U})\leq N_2\left(\tau,\frac{1}{\mathcal{U}}\right)+N_2(\tau,\mathcal{U})+N_2\left(\tau,\frac{1}{\mathcal{V}}\right)+N_2(\tau,\mathcal{V})+\frac{1}{2}\overline{N}\left(\tau,\frac{1}{\mathcal{U}}\right)+\overline{N_2}\left(\tau,\frac{1}{\mathcal{U}}\right)$$
$$+2\overline{N}(\tau,\mathcal{U})+\overline{N}\left(\tau,\frac{1}{\mathcal{V}}\right)+\overline{N}(\tau,\mathcal{V})\mathcal{S}(\tau,\mathcal{U})+\mathcal{S}(\tau,\mathcal{V}). \tag{50}$$

Make use of Lemma 5, we receive,

$$(n+m-2k)T(\tau,\varphi)\leq T\left(\tau,\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)+\mathcal{S}(\tau,\varphi) \tag{51}$$
$$\leq(k+1)(n+m)\,T(\tau,\varphi)+\mathcal{S}(\tau,\varphi).$$

$$(n+m-2k)T(\tau,\psi) \leq T\left(\tau,\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}\right)+\mathcal{S}(\tau,\psi) \leq (k+1)(n+m)\,T(\tau,\psi)+\mathcal{S}(\tau,\psi). \quad (52)$$

Since

$$T\left(\tau,\left(\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}\right) = dT\left(\tau,\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)+\ \mathcal{S}\left(\tau,\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right) \quad (53)$$

$$T\left(\tau,\left(\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}\right) = dT\left(\tau,\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}\right)+\ \mathcal{S}\left(\tau,\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}\right) \quad (54)$$

It is easy to see that

$$\mathcal{S}\left(\tau,\left(\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}\right) = \mathcal{S}\left(\tau,\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)+\ \mathcal{S}(\tau,\varphi)\ . \quad (55)$$

$$\mathcal{S}\left(\tau,\left(\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}\right) = \mathcal{S}\left(\tau,\left(\psi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)+\ \mathcal{S}(\tau,\psi) \quad (56)$$

On the other hand

$$N_2(\tau,\mathcal{U}) = N_2\left(\tau,\left(\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}\right) = 2\overline{N}(\tau,\varphi) \quad (57)$$

$$N_2(\tau,\mathcal{V}) = N_2\left(\tau,\left(\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}\right) = 2\overline{N}(\tau,\psi) \quad (58)$$

$$2\overline{N}(\tau,\mathcal{U}) = 2\overline{N}(\tau,\varphi) \quad (59)$$

$$2\overline{N}(\tau,\mathcal{V}) = 2\overline{N}(\tau,\psi) \quad (60)$$

By Lemma 6, we have

$$N_2\left(\tau,\frac{1}{\mathcal{U}}\right) = N_1\left(\tau,\frac{1}{\left(\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}}\right) = 2\overline{N}\left(\tau,\frac{1}{\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}}\right) \leq 2N_{k+1}\left(\tau,\frac{1}{\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)}\right)+2k\overline{N}\left(\tau,\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)+\ \mathcal{S}(\tau,\varphi) \quad (61)$$

$$\leq 2(k+1)\overline{N}\left(\tau,\frac{1}{\varphi}\right)+mN\left(\tau,\frac{1}{\varphi}\right)+2k\overline{N}(\tau,\varphi)+\mathcal{S}(\tau,\varphi).$$

$$2\overline{N}\left(\tau,\frac{1}{\mathcal{U}}\right) = 2\overline{N}\left(\tau,\frac{1}{\left(\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}}\right) = 2\overline{N}\left(\tau,\frac{1}{\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}}\right) \leq 2N_{k+1}\left(\tau,\frac{1}{\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)}\right)+2k\overline{N}\left(\tau,\frac{1}{\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)}\right)+\mathcal{S}\left(\tau,\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right) \quad (62)$$

$$\leq 2(k+1)\overline{N}\left(\tau,\frac{1}{\varphi}\right)+mN\left(\tau,\frac{1}{\varphi}\right)+2k\overline{N}(\tau,\varphi)+\mathcal{S}(\tau,\mathrm{f}).$$

$$\overline{N}\left(\tau,\frac{1}{\mathcal{V}}\right) = \overline{N}\left(\tau,\frac{1}{\left(\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}}\right) = \overline{N}\left(\tau,\frac{1}{\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}}\right) \quad (63)$$

$$\leq N_{k+1}\left(\tau,\frac{1}{\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)}\right)+k\overline{N}\left(\tau,\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)+\ \mathcal{S}\left(\tau,\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)$$

$$\leq 2(k+1)\overline{N}\left(\tau,\frac{1}{\psi}\right)+mN\left(\tau,\frac{1}{\psi}\right)+2k\overline{N}(\tau,\psi)+\mathcal{S}(\tau,\varphi).$$

$$N_2\left(\tau,\frac{1}{\mathcal{V}}\right) = 2\overline{N}\left(\tau,\frac{1}{\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}}\right)$$

$$\leq 2\left(\overline{N}\left(\tau,\frac{1}{\psi^{\mathrm{n+m-k}}}\right)\right)+N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}}\right) \quad (64)$$

$$\leq 2\left(\overline{N}\left(\tau,\frac{1}{\psi}\right)\right)+N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}}\right).$$

Combining (50), (53), (57), to (64), we deduce

$$T\left(\tau,\left(\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}\right)\leq(4k+2m+4)\overline{N}\left(\tau,\frac{1}{\varphi}\right)+(4k+4)\overline{N}(\tau,\varphi)+(k+m+3)\overline{N}\left(\tau,\frac{1}{\psi}\right)$$

$$+(k+2)\overline{N}(\tau,\psi)+2N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}}\right)+\mathcal{S}(\tau,\varphi)+\mathcal{S}(\tau,\psi)$$

$$\leq(6k+2m+8)T(\tau,\varphi)+(2k+m+5)T(\tau,\psi)+2k\overline{N}(\tau,\varphi)$$

$$+2N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}}\right)+\mathcal{S}(\tau,\varphi)+\mathcal{S}(\tau,\psi). \tag{65}$$

Likewise, we have

$$T\left(\tau,\left(\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)^{\mathrm{d}}\right)\leq(6k+2m+8)T(\tau,\psi)+(2k+m+5)T(\tau,\varphi)+2k\overline{N}(\tau,\psi)$$

$$+2N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}}\right)+\mathcal{S}(\tau,\varphi)+\mathcal{S}(\tau,\psi). \tag{66}$$

By Lemma 5, we have

$$(n+m-2k)dT(\tau,\varphi)+kdN(\tau,\varphi)+d\,N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}}\right)\leq dT\left(\tau,\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}\right)+\mathcal{S}(\tau,\varphi). \tag{67}$$

$$(n+m-2k)d\,T(\tau,\psi)+kdN(\tau,\psi)+d\,N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}}\right)\leq dT\left(\tau,\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}\right)+\mathcal{S}(\tau,\psi). \tag{68}$$

From (65), (66), (67) and (68), we have

$$(n+m-2k)dT(\tau,\varphi)+kdN(\tau,\varphi)+d\,N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}}\right)+(n+m-2k)dT(\tau,\psi)+kdN(\tau,\psi)$$

$$+d\,N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}}\right)\leq(6k+2m+8)T(\tau,\varphi)+(2k+m+5)T(\tau,\psi)+2k\overline{N}(\tau,\varphi)$$

$$+2N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}}\right)+(6k+2m+8)T(\tau,\psi)+4T(\tau,\psi)$$

$$+(2k+m+5)T(\tau,\varphi)+2k\overline{N}(\tau,\psi)+2N\left(\tau,\frac{\psi^{n+m-k}}{\left(\psi^{\mathrm{n}}\mathcal{Q}(\psi)\right)^{(\mathrm{k})}}\right)+\mathcal{S}(\tau,\varphi)+\mathcal{S}(\tau,\psi). \tag{69}$$

Since $d\geq 2$

$$dN\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}}\right)\geq 2N\left(\tau,\frac{\varphi^{n+m-k}}{\left(\varphi^{\mathrm{n}}\mathcal{Q}(\varphi)\right)^{(\mathrm{k})}}\right) \tag{70}$$

$$dN\left(\tau,\frac{\psi^{n+m-k}}{(\psi^{n}\mathcal{Q}(\psi))^{(k)}}\right)\geq 2N\left(\tau,\frac{\psi^{n+m-k}}{(\psi^{n}\mathcal{Q}(\psi))^{(k)}}\right) \tag{71}$$

$$kdN(\tau,\varphi)\geq 2k\overline{N}(\tau,\varphi) \tag{72}$$

$$kdN(\tau,\psi)\geq 2k\overline{N}(\tau,\psi) \tag{73}$$

Therefore

$$(nd+md-2kd-4k-2m-7)T(\tau,\varphi)+(nd+md-2kd-4k-2m-7)T(\tau,\psi)$$
$$\leq \mathcal{S}(\tau,\varphi)+\mathcal{S}(\tau,\psi). \tag{74}$$

which contradicts $n+m>2k+\frac{6k+2m+7}{.d}$. Hence $\mathcal{R}\equiv 0$. Similar to the argument in Theorem 3 we see that Theorem 4 holds.

Conclusion. Thesubjectofthepaperisthepowerofamero-morphicfúnctionwhosecer-taindifferentialpólynomialsharingasetwithitsderivative.Thetool,weusedinourdiscussion isclassicalvaluedistributiontheoryofmero-morphicfúnctións,whichisapartofthemilestóneofcómplexanálysisduringlastcentury.Hereweintroducetheideaofsetsharingproblemsin differentaspectsanditsrelevantconceptslikeuniquenessofdifferentialpólynomials.

Reference

[1] An,VuHoai;Khoai,HaHuy.Onuniquenessformero-morphicfúnctionsandtheir*n*thderivatives.Ann.Univ. Sci.Budapest.Sect.Comput.47,117–126,(2018).

[2] A.Banerjee,Meromorphicfúnctionssharingonevalue.Int.J.Math.Math.Sci.,no.22,3587–3598,2005

[3] Clunie,J.OnaresultofHayman.J.LondonMath.Soc.42,389–392,(1967).

[4] W.K.Hayman,*Meromorphicfúnctions*,OxfordMathematicalMonographs,ClarendonPress,Oxford,1964.

[5] W.K.Hayman,Researchproblemsinfúnctiontheory.TheAthlonePressUniversityofLondon,London,1967 vii+56pp.

[6] W.K.Hayman,Picardvaluesofmero-morphicfúnctionsandtheirderivatives, Ann.ofMath.(2)**70**,9–42,(1959).

[7] I.LahiandKalyanSinha.Lineardifferentialpólynomialssharingasetoftherootsofunity.Commun.Korean. Math.Soc.20,No.1,pp.1-10,2020.

[8] I.Lahiri,Weightedsharinganduniquenessofmero-morphicfúnctions,NagoyaMath.**J**.**161**,193–206,(2001).

[9] I.Lahiri,Weightedvaluesharinganduniquenessofmero-morphicfúnctions.ComplexVariablesTheoryAppl. 46,no.3,241–253,(2001).

[10] C.MengandX.Li,Onunicityofmero-morphicfúnctionsandtheirderivatives,**J**.Anal.**28**,no.3,879–894,(2020).

[11] Yang,Chung-Chun;Hua,Xinhou.Uniquenessandvalue-sharingofmero-morphicfúnctions.Ann.Acad.Sci. Fenn.Math.22,no.2,395–406,(1997).

[12] L. Yang,*Valuedistributiontheory*,translated and revised from the 1982 Chinese original,Springer-Verlag,Berlin,1993.

[13] Zhang,Qingcai.Meromorphicfúnctionthatsharingonesmallfúnctionwithitsderivative.**JIPAM**.**J**.Inequal. PureAppl.Math.6,no.4,Article116,13pp,(2005).

[14] S. Rajeshwari, Value distribution theory of Nevanlinna, Journal of Physics: conference series, 1597(1), 012 − 046.