Analytical and Numerical Solutions of Linear Volterra Integral Equations of the Second Kind with Weakly Singular Kernel by using the Sixth Order of Non_polynomial Spline Functions by Matlab

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Abstract

Volterra Integral Equations has wide range of the applications in physics and engineering problems.Spline function can be integrated and differentiated due to being piece wise polynomials and can easily store and implemented on digital computers, non-polynomial spline function apiece wise is a blend of trigonometric, as well as ,polynomial basis function ,which form a complete extended Chebyshev space.Matlab is a high-level software package with many built in functions that make the learning of numerical methods much easier and interesting.The aims of this paper is to focus and use the sixth order non-polynomial spline functions to solve linear Voltera integral equations with Weakly Singular Kernel. We followed the applied numerical method using Matlab .Numerical examples are presented to illustrate the applications of this method and to compare the computed results with other numerical methods for exact solutions.

Keywords: Linear Voltera integral equation, Weakly Singular Kernel, Non-polynomial Spline Functions, sixth order.

1.Introduction:

Recently there is increase in concern by integrated equations, particularly Voltera equations in several fields such as the theory of hypothesis and the issues of Dirkhelia in static electricity and the problems of astrophysics. In the mid of 1960s the numerical solution of integral equations was begun when the Kernel is unwell . In the beginning of 1980s the numerical methods began to take more importance when there is no analytical solution, particularly when the kernel is weakly singular.

2.Non-polynomial Spline Functions Method:

In 1997 Diogo, T and Lima, B presented deductive method [10], in order to find a numerical solution for Voltera integral equations with weakly singular kernel. In 2012

Hossinpour presented [8] a solution for differential integral equations by using non-polynomial spline functions. In 2015, Harbis,S , Murad,M and Majed,S[7] presented a numerical solution for linear Voltera integral equation from second kind by using non-polynomial spline function from the third degree. In 2017, Najwa, S and Mohammed, S[12] presented a numerical solution for linear Voltera integral equation from second kind by using non-polynomial spline function from the fifth degree.

In this paper we introduced a numerical solution for linear Voltera integral equation from second kind with weakly singular kernel by using non-polynomial spline function from the sixth degree by Matlab .

3.Voltera Integral Equations

Definition(3.1):[13]

The most standard form of Voltera integral equation is given as [14]

$$Q(x)u(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)u(t)dt, \qquad (1)$$

Where $a(x)$and $b(x)$ are the limits of the integration, λ is a constant parameter, and $k(x,t)$ is a function of two variables x,t called the kernel or the nucleus of the integral equation.

An integral equation (1) is called non-linear integral equation, if the kernel $k(x,t)$ is given in the form$k(x,t,u(t))$.

An integral equation (1) is called homogeneous if $f(x) = 0$ and otherwise nonhomogeneous [1].

An integral equation (1) is called linear of first kind if $Q(x) = 0$ and is called linear of the second kind if $Q(x) = 1$.

An integral equation (1) is called Voltera integral equation if $a(x) = a$ and $b(x) = x$ then equation (1.1) becomes [12],[13]

$$Q(x)u(x) = f(x) + \lambda \int_{a}^{x} k(x,t)u(t)dt, x \in [a,b] \qquad (2)$$

An integral equation (2) is called Voltera integral equation of the first kind if $Q(x) = 0$ [11,13].

An integral equation (2) is called Voltera integral equation of the second kind if $Q(x) = 1$.

The integral equation (2) is called Voltera integral equation is not first kind and second kind when $Q(x)$ is neither 0 nor 1.

If the kernel integral equation (1) depends on the difference $(x - t)$, then it called difference kernel and the equation is called integral equation of convolution type .i.e.,

$$K(x,t) = k(x-t) \qquad (3)$$

Here we can apply Laplace transform to get the exact solution.

The kernel is called degenerate or (sparable) kernel, when the kernel may be decompose as follows:[9]

$$k(x,t) = \sum_{k=1}^{n} a_k(x)b_k(t) \qquad (4)$$

If the kernel $k(x,t)$ is in the form [13]

$$k(x,t) = \frac{H(x,t)}{(x-\varepsilon)^{\alpha}} \qquad (5)$$

Where H is bounded in $D = a \le x \le b \; and \; a \le t \le b$ with

$H(x,t) \neq 0$ Andα is constant s.t $0 \leq \alpha \leq 1$ then the integral equation (1) is called weakly singular kernel
The equation of the form:

$$f(x) = \int_0^x \frac{u(x)}{(x-t)^\alpha} dt \quad 0 \leq \alpha \leq 1 \qquad (6)$$

Or of the second kind

$$u(x) = f(x) + \int_0^x \frac{u(x)}{(x-t)^\alpha} dt \quad 0 \leq \alpha \leq 1 \qquad (7)$$

Are called generalized Abel's integral equation and weakly singular integral equation respectively.
for

$$f(x) = \int_0^x \frac{u(x)}{(x-t)^{\frac{1}{2}}} dt \qquad (8)$$

It called the Abel's singular integral equation.
We focus our concern on the equation of the form:

$$u(x) - \int_0^t \frac{t^{m-1}}{x^m} u(t)dt = f(x), x \in [0,T] \qquad (9)$$

, Which is Voltera integral equation of the second kind with weakly singular kernel. Where$u(t)$is unknown function and f is known function. Where $0 < \mu < 1$.However there is a singularity at $t = 0$ and $s = 0$ for any positive value of t.

4. Exact Solution of Linear VIE's of the Second Kind with Weakly Singular Kernel [10, 11, 13]:

We consider the second kind of VIE's with weakly singular kernel (9)[5]

$$u(x) - \int_0^t \frac{t^{m-1}}{x^m} u(t)dt = f(x), x \in [0,T]$$

Where $0 < \mu < 1$ and f is known function. There is a singularity at $t = 0$ and $s = 0$ for any positive value of t.

In [10] the author gives suggestion for the analytic solution to solve linear VIE's of the second kind with weakly singular kernel.

a) If $0 < \mu < 1$ and $f \in C^{(1)}[0,t]$ (with $f(0) = 0$ if $\mu = 1$) then VIE's of the second kind with weakly singular kernel (9)has many solution $u \in C[0,t]$

$$u(t) = C_0 t^{1-\mu} + f(t) + \gamma + t^{1-u} \int_0^t s^{\mu-2}\big(f(s) - f(0)\big)ds \qquad (10)$$

Where

$$\gamma = \begin{cases} \frac{1}{\mu - 1} f(0) & , if\ \mu < 1 \\ 0 & if\ \mu = 1 \end{cases} \qquad (11)$$

And C_0 is an arbitrary constant.

b) If $\mu > 1$ and $f \in C^m[0,t], \mu \geq 0$,then the unique solution $u \in C^m[0,t]$is :

$$u(t) = f(t) + t^{1-\mu} \int_0^t s^{\mu-2} f(s)ds, \qquad (12)$$

5. A Numerical Solution of Linear Voltera Integral Equations of Second Kind with Weakly Singular Kernel using the Sixth Order of Non-Polynomial Spline Functions

The form of the sixth non-polynomial spline function is:

$$P_i(t) = a_i cosK(t-t_i) + b_i sinK(t-t_i) + c_i(t-t_i) + d_i(t-t_i)^2 + e_i(t-t_i)^3 + r_i(t-t_i)^4 + z_i(t-t_i)^5 + g_i(t-t_i)^6 + q_i \qquad (13)$$

$where$ $a_i, b_i, c_i, d_i, e_i, r_i, z_i g_i, and$ $q_i$areconstants, to be determined .In order to obtain the values of $a_i, b_i, c_i, d_i, e_i, r_i, z_i, g_i and$ q_i,we differentiate equation (13) eight times with respect to t ,and we get the following equations:

$$\boldsymbol{P_i^{(1)}(t)} = -Ka_i sinK(t-t_i) + Kb_i cosK(t-t_i) + c_i + 2d_i(t-t_i) + 3e_i(t-t_i)^2 + 4r_i(t-t_i)^3 + 5z_i(t-t_i)^4 + 6g_i(t-t_i)^5$$

$$\boldsymbol{P_i^{(2)}(t)} = -K^2 a_i cosK(t-t_i) - K^2 b_i sinK(t-t_i) + 2d_i + 6e_i(t-t_i) + 12r_i(t-t_i)^2 + 20z_i(t-t_i)^3 + 30g_i(t-t_i)^4$$

$$\boldsymbol{P_i^{(3)}(t)} = K^3 a_i sinK(t-t_i) - K^3 b_i cosK(t-t_i) + 6e_i + 24r_i(t-t_i) + 60z_i(t-t_i)^2 + 120g_i(t-t_i)^3$$

$$\boldsymbol{P_i^{(4)}(t)} = K^4 a_i cosK(t-t_i) + K^4 b_i sinK(t-t_i) + 24r_i + 120z_i(t-t_i) + 360g_i(t-t_i)^2$$

$$\boldsymbol{P_i^{(5)}(t)} = -K^5 a_i sinK(t-t_i) + K^5 b_i cosK(t-t_i) + 120z_i + 720g_i(t-t_i)$$

$$\boldsymbol{P_i^{(6)}(t)} = -K^6 a_i cosK(t-t_i) - K^6 b_i sinK(t-t_i) + 720g_i$$

$$\boldsymbol{P_i^{(7)}(t)} = K^7 a_i sinK(t-t_i) - K^7 b_i cosK(t-t_i)$$

$$\boldsymbol{P_i^{(8)}(t)} = K^8 a_i cosK(t-t_i) + K^8 b_i sinK(t-t_i)$$

$$\forall \; equations \; (14)$$

Hence replace t by t_i in the relation (13) and (14) yields:

$$P(t) = a_i + q_i$$

$$P_i^{(1)}(t) = b_i K + c_i$$

$$P_i^{(2)}(t) = -K^2 a_i + 2d_i$$

$$P_i^{(3)}(t) = -K^3 b_i + 6e_i$$

$$P_i^{(4)}(t) = K^4 a_i + 24r_i$$

$$P_i^{(5)}(t) = K^5 a_i + 120z_i$$

$$P_i^{(6)}(t) = -K^6 a_i + 720\mathrm{g_i}$$

$$P_i^{(7)}(t) = -K^7 b_i$$

$$P_i^{(8)}(t) = K^8 a_i$$

We obtain the values of$a_i, b_i, c_i, d_i, e_i, r_i, z_i, g_i, and$ q_i from the above relations as follows:

$$a_i = \frac{1}{K^8} P_i^{(8)}(t) \qquad (15)$$

$$b_i = -\frac{1}{K^7} P_i^{(7)}(t) \qquad (16)$$

$c_i = P_i^{(1)}(t) - Kb_i$ (17)

$d_i = \frac{1}{2}[P_i^{(2)}(t) + K^2 a_i]$ (18)

$e_i = \frac{1}{6}[P_i^{(3)}(t) + K^3 b_i]$ (19)

$r_i = \frac{1}{24}[P_i^{(4)}(t) - K^4 a_i]$ (20)

$z_i = \frac{1}{120}[P_i^{(5)}(t) - K^5 b_i]$ (21)

$g_i = \frac{1}{720}[P_i^{(6)}(t) + K^6 b_i]$ (22)

$q_i = P_i(t) - Ka_i$ (23)

$for\ i = 0,1, \dots, n$

6. Linear VIE's of the Second Kind with Weakly Singular Kernel:

Let $u(x) - \int_0^x \frac{t^{\mu-1}}{x^\mu} u(t)dt = f(x), x \in [0, T]$(24)

Where $0 < \mu < 1$ and $f$is known function.

So to solve (24), we multiply theboth two sides of (24) by x^μ yields to :

$$x^\mu u(x) - \int_0^t t^{\mu-1}\, u(t)dt = f(x)x^\mu \qquad (25)$$

Hence differentiation (25) with respect to x,we get:

$$x^\mu u^{(1)}(x) + \mu x^{\mu-1} u(x) - \frac{1}{x^{1-\mu}} u(x) = \mu x^{\mu-1} f(x) + f^{(1)}(x) x^\mu \qquad (26)$$

And multiplication the both two sides of (26) by $x^{1-\mu}$ yields:

$$xu^{(1)}(x) + (\mu - 1)u(x) = \mu f(x) + xf^{(1)}(x) \qquad (27)$$

Where

$$u_0 = \frac{\mu}{\mu - 1} f(0) \qquad (28)$$

Hence differentiate equation (27) eight times with respect to x we get:

$$xu^{(2)}(x) + \mu u^{(1)}(x) = (\mu + 1)f^{(1)}(x) + xf^{(2)}(x)$$
$$xu^{(3)}(x) + (\mu + 1)u^{(2)}(x) = (\mu + 2)f^{(2)}(x) + xf^{(3)}(x)$$
$$xu^{(4)}(x) + (\mu+2)u^{(3)}(x) = (\mu + 3)f^{(3)}(x) + xf^{(4)}(x)$$
$$xu^{(5)}(x) + (\mu + 3)u^{(4)}(x) = (\mu + 4)f^{(4)}(x) + xf^{(5)}(x)$$
$$xu^{(6)}(x) + (\mu + 4)u^{(5)}(x) = (\mu + 5)f^{(5)}(x) + xf^{(6)}(x)$$
$$xu^{(7)}(x) + (\mu + 5)u^{(6)}(x) = (\mu + 6)f^{(6)}(x) + xf^{(7)}(x)$$
$$xu^{(8)}(x) + (\mu + 6)u^{(7)}(x) = (\mu + 7)f^{(7)}(x) + xf^{(8)}(x)$$
$$xu^{(9)}(x) + (\mu + 7)u^{(8)}(x) = (\mu + 8)f^{(8)}(x) + xf^{(9)}(x)$$
$$\forall\ equations\ (29)$$

Hence replace $x\ by\ a$ in the relation (29), yields:

$$u_0^{(1)} = \frac{\mu + 1}{\mu} f^{(1)}(a)$$

$$u_0^{(2)} = \frac{\mu + 2}{\mu + 1} f^{(2)}(a)$$

$$u_0^{(3)} = \frac{\mu + 3}{\mu + 2} f^{(3)}(a)$$

$u_0^{(4)} = \frac{\mu+4}{\mu+3} f^{(4)}(a)$
$u_0^{(5)} = \frac{\mu+5}{\mu+4} f^{(5)}(a)$
$u_0^{(6)} = \frac{\mu+6}{\mu+5} f^{(6)}(a)$
$u_0^{(7)} = \frac{\mu+7}{\mu+6} f^{(7)}(a)$
$u_0^{(8)} = \frac{\mu+8}{\mu+7} f^{(8)}(a)$

$$\forall\ equations \quad (30)$$

Algorithm (VIE2WSKNPS6):

Algorithm of Voltera integral equations with weakly singular kernel using the sixth order of non-polynomial spline functions:

Step 1: set

$$h = \frac{b-a}{n}, \quad t_i = t_0 + ih\ , i = 0,1, \dots., n$$
$$where\ t_0 = a, t_n = b\ and\ u_0 = \frac{\mu}{\mu-1} f(a).$$

Step 2: Evaluate $a_0, b_0, c_0, d_0, e_0, r_0, z_0\ , g_i and\ q_0$by substituting (28) and (30) In equations (15) - (23).

Step 3: Calculate $p_0(t)$using step 2 and equation (13).

Step 4: Approximate $u_1 \approx p_0(t_i)$.

Step 5: For $i = 1\ to\ n - 1$ do following steps:

Step 6: Evaluate $a_i, b_i, c_i, d_i, e_i, r_i\ and\ g_i$substituting in equations (15)-(23) and replacing $u(t)$and it's derivatives by$p_i(t)$ and its derivatives.

Step 7: Calculate $p_i(t)$using step 6 and equation (13).

Step 8: Approximate $u_{i+1} = p_i(t_{i+1})$.

7. Programming:

Program 1: linear non-polynomial spline function for solving VIE's of the second kind with weakly singular kernel.

```
function [X, Y, err, u]=singular(f,ex,LB,UB,n,m)
syms X
h = (UB-LB)/(n - 1);
X = LB:h:UB;
u(1)=(m/(m-1))*subs(f,0);
du(1)=((m+1)/m)*subs(diff(f,1),0);
d2u(1)=((m+2)/(m+1))*subs(diff(f,2),0);
d3u(1)=((m+3)/(m+2))*subs(diff(f,3),0);
d4u(1)=((m+4)/(m+3))*subs(diff(f,4),0);
a(1)=-d2u(1);
b(1)=-d3u(1);
c(1)=du(1)-b(1);
d(1)=u(1)-a(1);
for i=1:n-1
```

```
   u(i+1)=a(i)*cos(h)+b(i)*sin(h)+h*c(i)+d(i);
   du(i+1)=-a(i)*sin(h)+b(i)*cos(h)+c(i) ;
d2u(i+1)=-a(i)*cos(h)-b(i)*sin(h);
d3u(i+1)=a(i)*sin(h)-b(i)*cos(h) ;
d4u(i+1)=a(i)*cos(h)+b(i)*sin(h) ;
a(i+1)=-d2u(i+1);
b(i+1)=-d3u(i+1);
c(i+1)=du(i+1)-b(i+1);
d(i+1)=u(i+1)-a(i+1);
end
Y = subs(ex, X);
for i=1:n
err(i)=abs(u(i)-subs(ex,X(i)));
end
format long
disp('    i                 X                  Y                 u                 err')
[1:n; X; Y; u; err]'
figure(2)
plot(X, Y, 'b --', 'linewidth', 3)
hold on
plot(X, u, 'r .-', 'linewidth', 3)
grid on
xlabel('X')
ylabel('Y')
title('exact vs. approx.')
legend('exact', 'approx.')
```

Program 2: sixth non –polynomial spline function for solving VIE's of the second kind with weakly singular kernel.

```
function [X, Y, err, u]=singular(f,ex,LB,UB,n,m)
syms X
h = (UB-LB)/(n - 1);
X = LB:h:UB;
u(1)   = (m/(m - 1))*subs(f, 0);
du(1)  = ((m + 1)/m)*subs(diff(f, 1), 0);
d2u(1) = ((m + 2)/(m + 1))*subs(diff(f, 2), 0);
d3u(1) = ((m + 3)/(m + 2))*subs(diff(f, 3), 0);
d4u(1) = ((m + 4)/(m + 3))*subs(diff(f, 4), 0);
d5u(1) = ((m + 5)/(m + 4))*subs(diff(f, 5), 0);
d6u(1) = ((m + 6)/(m + 5))*subs(diff(f, 6), 0);
d7u(1) = ((m + 7)/(m + 6))*subs(diff(f, 7), 0);
d8u(1) = ((m + 8)/(m + 7))*subs(diff(f, 8), 0);
a(1) = d8u(1);
b(1) = -d7u(1);
```

```
c(1) = du(1) - b(1);
d(1) = (1/2)  *(d2u(1) + a(1));
e(1) = (1/6)  *(d3u(1) + b(1));
r(1) = (1/24 )*(d4u(1) - a(1));
z(1) = (1/120)*(d5u(1) - b(1));
g(1) = (1/720)*(d6u(1) + a(1));
q(1) = u(1) - a(1);
for i =1:n-1
   u(i+1)   =  a(i)*cos(h) + b(i)*sin(h) + h*c(i)   + d(i)*h^2   + e(i)*h^3     + r(i)*h^4     +
z(i)*h^5    + g(i)*h^6    + q(i);
   du(i+1)  = -a(i)*sin(h) + b(i)*cos(h) + c(i)     + 2*d(i)*h   + 3*e(i)*h^2   + 4*r(i)*h^3   +
5*z(i)*h^4  + 6*g(i)*h^5;
   d2u(i+1) = -a(i)*cos(h) - b(i)*sin(h) + 2*d(i)   + 6*e(i)*h   + 12*r(i)*h^2  + 20*z(i)*h^3
+ 30*g(i)*h^4;
   d3u(i+1) =  a(i)*sin(h) - b(i)*cos(h) + 6*e(i)   + 24*r(i)*h  + 60*z(i)*h^2  + 120*g(i)*h^3;
   d4u(i+1) =  a(i)*cos(h) + b(i)*sin(h) + 24*r(i)  + 120*z(i)*h + 360*g(i)*h^2;
d5u(i+1) = -a(i)*sin(h) + b(i)*cos(h) + 120*z(i) + 720*g(i)*h;
d6u(i+1) = -a(i)*cos(h) - b(i)*sin(h) + 720*g(i);
d7u(i+1) =  a(i)*sin(h) - b(i)*cos(h);
d8u(i+1) =  a(i)*cos(h) + b(i)*sin(h);
a(i+1) = d8u(i+1);
b(i+1) = -d7u(i+1);
c(i+1) = du(i+1) - b(i+1);
d(i+1) = (1/2)*(d2u(i+1) + a(i+1));
e(i+1) = (1/6)*(d3u(i+1) + b(i+1));
r(i+1) = (1/24)*(d4u(i+1) - a(i+1));
z(i+1) = (1/120)*(d5u(i+1) - b(i+1));
g(i+1) = (1/720)*(d6u(i+1) + a(i+1));
q(i+1) = u(i+1) - a(i+1);
end
Y = subs(ex, X);
for i=1:n
err(i)=abs(u(i)-subs(ex,X(i)));
end
format long
disp('    i              X                Y               u                err')
[1:n; X; Y; u; err]'
figure(1)
plot(X, Y, 'b --', 'linewidth', 3)
hold on
plot(X, u, 'r .-', 'linewidth', 1)
grid on
xlabel('X')
ylabel('Y')
```

title('exact vs. approx.')
legend('exact', 'approx.')

8. Numerical Examples:

Example(8.1): Consider the VIE of Second Kind with Weakly Singular Kernel[12.14]:

$$u(x) - \int_0^x \frac{t^{m-1}}{x^m} u(t)dt = f(x), \qquad 0 \le x \le 1$$

Where $f(x) = x + 1$ $\quad and \quad$ $\mu = 0.5,$ $\quad with\, u(x) = \frac{\mu}{\mu-1} + \frac{\mu+1}{\mu} x$

Table(8.1) present a comparison between the exact and numerical solution using linear and sixth order non-polynomial spline functions, where $u_i(x)$ denote the approximate solution non- polynomial spline functions , with $h = 0.1$.

Table (8.1):Exact and numerical solution of test example(8.1)

x	Exact solution	$u_i(x)$	
		linear	Sixth order
0	-1.00000000000000	-1.00000000000000	-1.00000000000000
0.1	-0.70000000000000	-0.70000000000000	-0.70000000000000
0.2	-0.40000000000000	-0.40000000000000	-0.40000000000000
0.3	-0.10000000000000	-0.10000000000000	-0.10000000000000
0.4	0.200000000000000	0.200000000000000	0.200000000000000
0.5	0.500000000000000	0.500000000000000	0.500000000000000
0.6	0.800000000000000	0.800000000000000	0.800000000000000
0.7	1.100000000000000	1.100000000000000	1.100000000000000
0.8	1.400000000000000	1.400000000000000	1.400000000000000
0.9	1.700000000000000	1.700000000000000	1.700000000000000
1	2.000000000000000	2.000000000000000	2.000000000000000

Table (8.2): present a comparison between the error in our methods and other method in []where error=|exact value-numerical|and $\|err\| = max|error|$

Table (8.2):comparison between the error with

x	Error in linear	Error in sixth order
0	0.000000000000000	0.000000000000000
0.1	0.000000000000000	0.000000000000000
0.2	0.000000000000000	0.000000000000000
0.3	0.000000000000000	0.000000000000000
0.4	0.000000000000000	0.000000000000000
0.5	0.000000000000000	0.000000000000000
0.6	0.000000000000000	0.000000000000000
0.7	0.000000000000000	0.000000000000000
0.8	0.000000000000000	0.000000000000000
0.9	0.000000000000000	0.000000000000000
1	0.000000000000000	0.000000000000000
$\|err\|_\infty$	0.000000000000000	0.000000000000000

Example(8.2): Consider the VIE of second kind with Weakly Singular kernel[12,14]:

$$u(x) - \int_0^x \frac{t^{m-1}}{x^m} u(t)dt = f(x), \qquad 0 \le x \le 1$$

Where $f(x) = x^2 + x + 1$ and $\mu = 0.5$, $with\, u(x) = \frac{\mu}{\mu-1} + \frac{\mu+1}{\mu}x + \frac{\mu+2}{\mu+1}x^2$

Table(1) present a comparison between the exact and numerical solution using linear and sixth order non-polynomial spline functions, where $u_i(x)$ denote the approximate solution non- polynomial spline functions , with $h = 0.1$.

Table (8.3):Exact and numerical solution of test example(8.2)

x	Exact solution	$u_i(x)$	
		linear	Sixth order
0	-1.00000000000000	-1.00000000000000	-1.00000000000000
0.1	-0.68333333300000	-0.68334721759342	-0.68333333333333
0.2	-0.33333333200000	-0.33355525947081	-0.33333333333333
0.3	0.05000000300000	0.04887836958131	0.05000000000000
0.4	0.46666667200000	0.46313001999038	0.46666666666667
0.5	0.91666667500000	0.90805812703209	0.91666666666667
0.6	1.40000001200000	1.38221461696774	1.40000000000000
0.7	1.91666668300000	1.88385937571837	1.91666666666667
0.8	2.46666668800000	2.41097763550945	2.46666666666667
0.9	3.05000002700000	2.96130010576445	3.05000000000000
1	3.66666670000000	3.53232564710620	3.66666666666667

Table (8.4): present a comparison between the error in our methods and other method in [ ] where error=|exact value - numerical| and $\|err\|_\infty = max|error|$

Table (8.4):comparison between the error with

x	Error in linear	Error in sixth order
0	0.00000000000000	0.00000000000000
0.1	0.00001388459342	0.00000000000333
0.2	0.00022192747081	0.00000000001333
0.3	0.00112163341869	0.00000000003000
0.4	0.00353665200962	0.00000000005333
0.5	0.00860854796791	0.00000000008333
0.6	0.01778539503226	0.00000000012000
0.7	0.03280730728163	0.00000000016333
0.8	0.05568905249055	0.00000000021333
0.9	0.08869992123555	0.00000000027000
1	0.13434105289380	0.00000000033333
$\|err\|_\infty$	0.13434105289380	0.00000000033333

9.Figures:

Fig (9.1): Comparison between the analytical and the approximate solution using linear and sixth order non-polynomial spline functions for example (8.1)

Figure for the example(8.1) : the linear

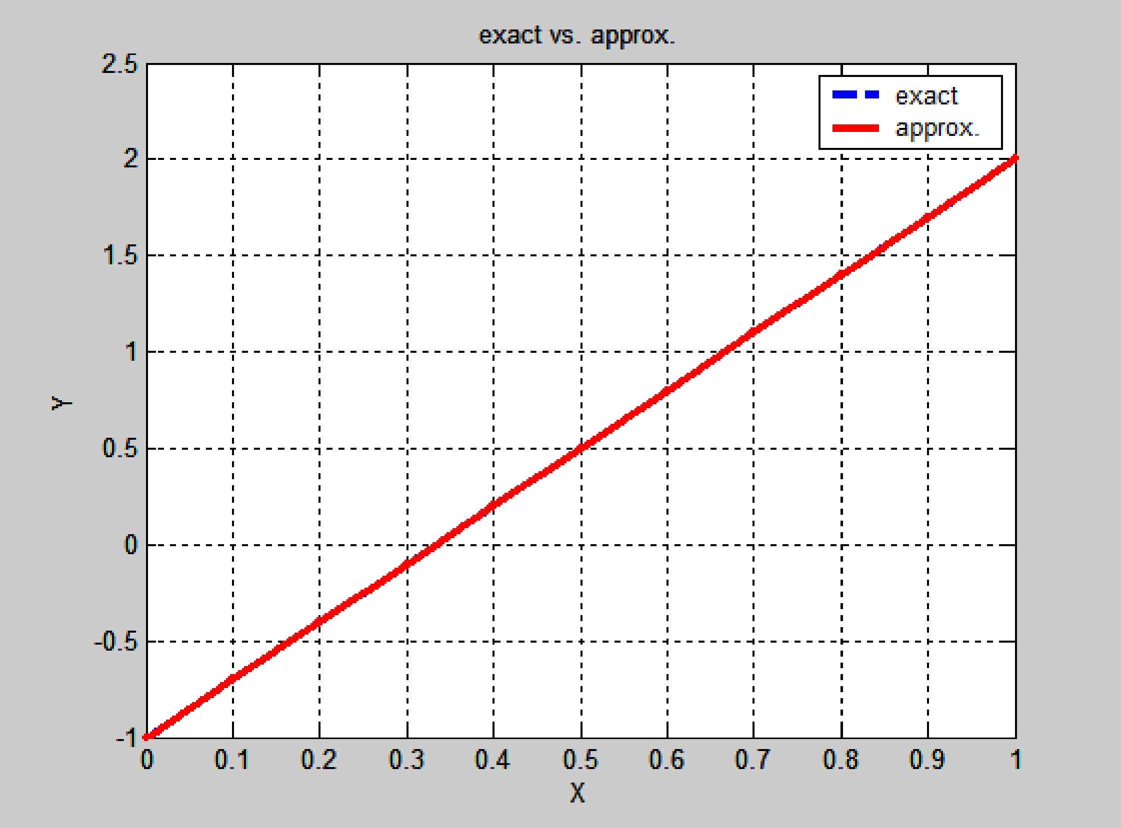


Figure for theexample(8.1):sixth order

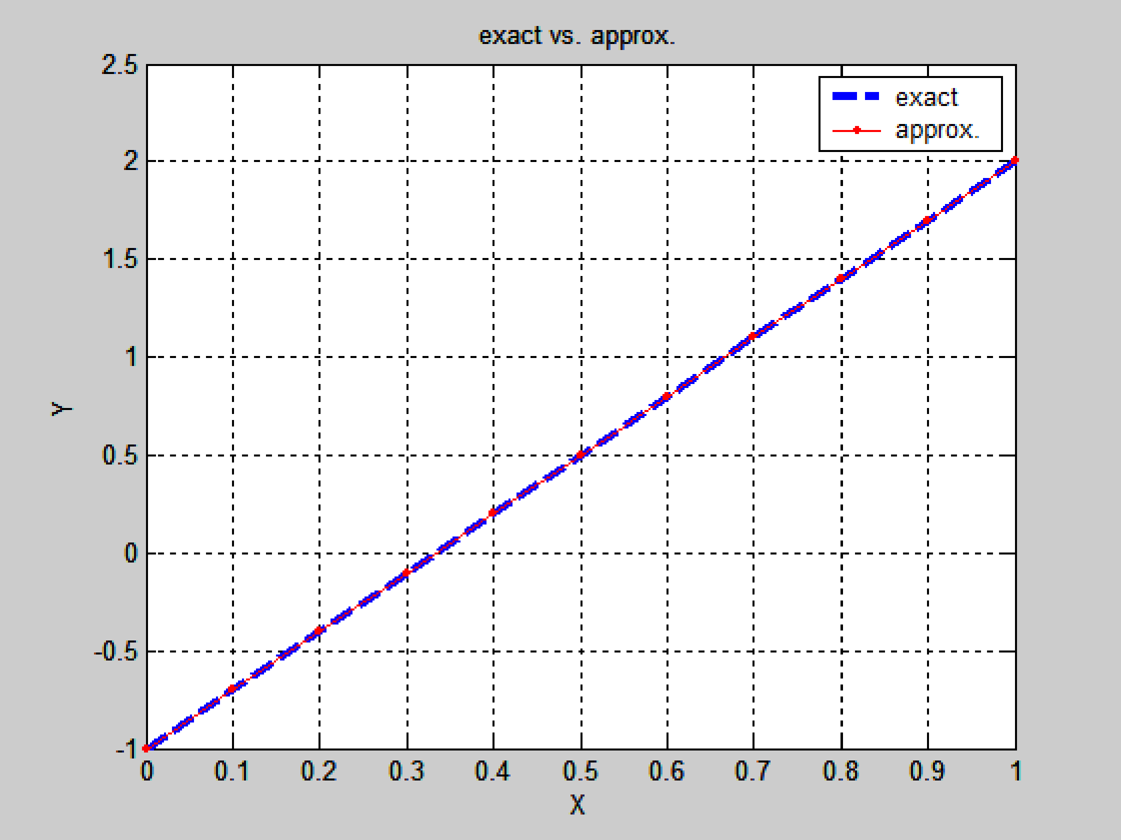


Fig (9.2): Comparison between the analytical and the approximate solution using linear and sixth order non-polynomial spline functions for example (8.2)

Figure for theexample(8.2): the linear

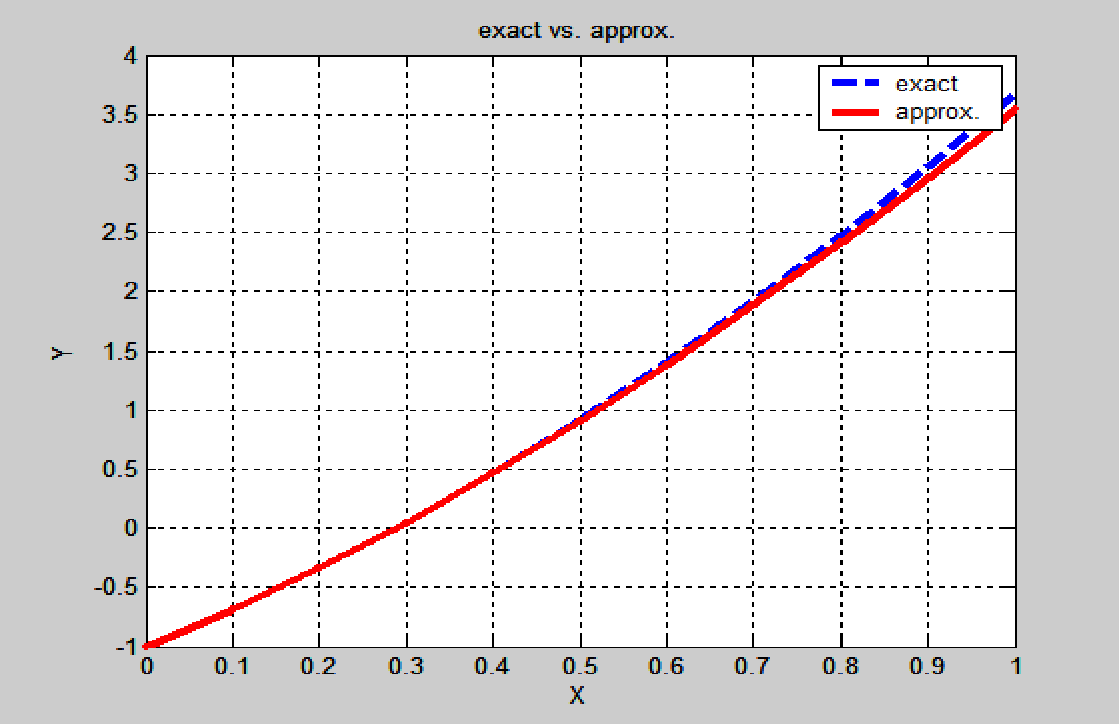
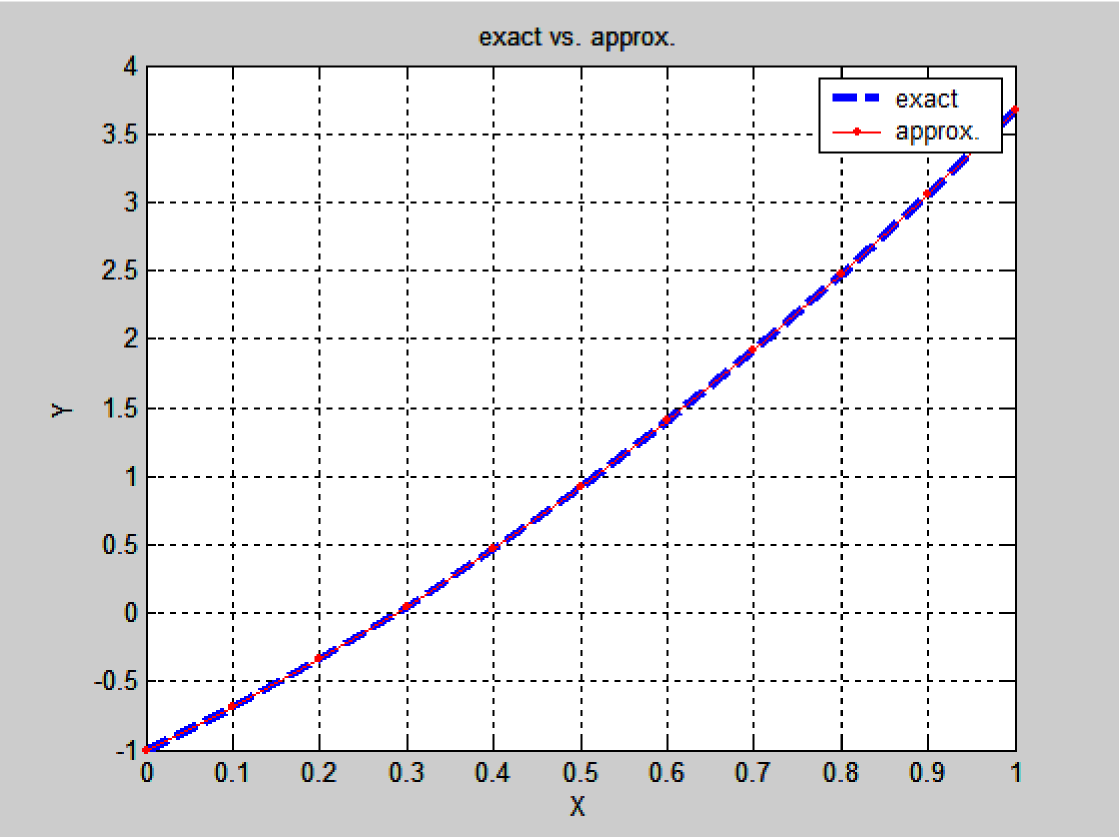


Fig for the example(8.2) :the sixth order



10.Results:

In general the methods which we used in this paper , proved their effectiveness in solve linear Volterra Integral Equations with weakly singular kernel numerically and finding an accurate results and the figures(9.2) show a comparison between the analytical and numerical solution which was presented in example(8.2) ,the results show that the sixth order non- polynomial Spline functions gives the best approximation from the linear non-polynomial Spline functions.

Conclusion:

Finally the results that are obtained in our work show that the sixth order non-polynomial Spline functions gives the best approximation to solve from the linear non-polynomial Spline functions , the sixth order non- polynomial Spline functions characterized with an easy, fast and accuracy is high comparing with the less sixth order non- polynomial Spline functions and the proposed scheme is simple and computationally attractive and its accuracy is high and we can simply execute this method in Matlab.

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