A Note on the Numbers Constructed by Fibonacci Numbers

Rajiv Kumar[a], Satish Kumar[b] and M. K. Sharma[c]
[a] Department of Mathematics, D. J. College, Baraut (Baghpat); India
[b] Department of Mathematics, D. N. (PG) College, Meerut; India
[c] Department of Mathematics, Chaudhary Charan Singh University, Meerut-250004, India
Email: [a] rajiv73kr@gmail.com & Orcid id:- https://orcid.org/0000-0001-7601-3279),
[b]skg22967@gmail.com,
[c]drmukeshsharma@gmail.com & Orcid id:-https://orcid.org/0000-0003-3071-5931

Abstract: In this research paper we have obtained some numerical and generalized results for the numbers of the form $N_k(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^k$, $k \geq 1$. Where all numbers $N_k(F_{i+1})$ is constructed by using the Fibonacci numbers F_{i+1} , 1≤i≤4. Here we found out that the numbers of the form $N_k(F_{i+1})$ are static nature when the value of k is even and dynamic nature when the value of k is odd.

Keywords: Even Number, Fibonacci number, Odd Number, Prime Number, Fibonacci sequence

1. Introduction

It is well known that Fibonacci sequence is wonderful and amazing creations in number theory. The Fibonacci sequence is the best known work of Leonard of Pisa (Fibonacci). The terms of Fibonacci sequence are known as Fibonacci numbers. In Fibonacci sequence each new term (number) is the sum of the two terms (numbers) preceding it **(Delvin, K., 2011)**. Many researchers and mathematicians have been studying Fibonacci numbers in many different forms for centuries. There are multitudes of properties of Fibonacci numbers discussed by **(Garland, 1987)**, **(Posamentier & Lehmann, 2007)**.If we recall such properties of Fibonacci sequences as well as Fibonacci numbers from mid-18th century to till now, then we find a huge collection of results regarding Fibonacci numbers (terms of Fibonacci sequences). Some properties are as follows as; any two consecutive Fibonacci numbers are relatively prime **(Garland, 1987 page- 67)**. Every third Fibonacci number is divisible by $F_3 = 2$. Every fourth Fibonacci number is divisible by $F_4 = 3$. Every fifth Fibonacci number is divisible by $F_5 = 5$, and so on. In general, every nth Fibonacci number is divisible by the nth term in the Fibonacci sequence (page 69). Multiplying any Fibonacci number by two and subtracting the next number in sequence will give the result as $2\,F_n - F_{n+1} = F_n - 2$(page 70), etc. **(David, M., Burton, 2010; Garland, 19870)**. Also summing together any ten consecutive Fibonacci numbers will always be divisible by eleven (page 33). Composite number positions Fibonacci numbers are always composite excluding fourth Fibonacci number (page 35), etc. **(Posamentier & Lehmann, 2007).**In the present paper authors are motivated by the wonderful and amazing properties of Fibonacci numbers discovered by researchers and mathematicians as per the study of sufficient literature available in references(**Atanasov, K., David, M. Burton, …………., T. Koshy).**

In this research paper we consider a class of numbers of the form $N\,_k(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^k$ $,k \geq 1$, those are constructed by using the Fibonacci numbers $F_{i+1}, 1 \leq i \leq 4$, those are the members of classical Fibonacci sequence $F\ _0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}, n \geq 2$. In this research paper, we have approached the theory of congruence to find out the nature of the numbers $N_k(F_i)$ for all k. Also we consider every integer $k$is of the form $4n$, $4n + 1$, $4n2 + 2$, and $4n + 3$.

2. Theorem: If $k = 4n$, then prime number 3 must be divide$N_k(F_{i+1})$, $\forall k \geq 1$.

Proof: We have

$N_k(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^k$ $,k \geq 1$.

Let $k = 4n,\ n = 1,2, \dots \dots \dots$ then-

$$N_{k=4n}(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^{4n}$$

clearly we have-

$\sum_{i=1}^{4} {F_{i+1}}^{4n} \equiv 0\ (mod\ 3)$.

The congruence relation implies that $3 \mid N_k(F_{i+1})$.

3. **Theorem: If $k = 4n + 2$, then prime number 3 must be divide $N_k(F_{i+1})$, $\forall k \geq 1$.**

Proof: We have

$N_k(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^k , k \geq 1$.

Let $k = 4n + 2$, $n = 0, 1, 2, \dots \dots \dots$ then-

$$N_{k=4n+2}(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^{4n+2}$$

clearly we have-

$\sum_{i=1}^{4} {F_{i+1}}^{4n+2} \equiv 0 \ (mod\ 3)$.

It implies that $3 \mid N_k(F_{i+1})$.

4. **Theorem: If $k = 4n + 1$, then numbers of the form $N_k(F_{i+1})$ may or may not be divisible by some prime number.**

Proof: We have

$N_k(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^k \ , k \geq 1$.

Let $k = 4n + 1$, $n = 0, 1, 2, \dots \dots \dots$ then-

$$N_{k=4n+1}(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^{4n+1}$$

If we take $n = 0$ then $k = 1$, clearly-

$$N_{k=1}(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^1 = 11$$

which is a prime number.

Again if we take $n = 1$ then k will be 5, clearly we have-

$$N_{k=5}(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^5 \equiv 0 \ (mod\ 19)$$

In the same manner if we take $n = 4$ then k will be 5, clearly we have-

$$N_{k=17}(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^{17} = 763{,}068{,}724{,}361$$

This is also a prime number.

All congruence shows that- If $k = 4n + 1$, then numbers of the form $N_k(F_{i+1})$ may or may not be divisible by some prime number.

5. **Theorem: If $k = 4n + 3$, then numbers of the form $N_k(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^k$ may be divisible by some prime number.**

Proof: We have

$N_k(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^k \ , k \geq 1$.

Let $k = 4n + 3$, $n = 0, 1, 2, \dots \dots \dots$ then-

$$N_{k=4n+3}(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^{4n+3}$$

If we take $n = 0$ then $k = 3$, clearly-

$$N_{k=3}(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^{3} \equiv 0\ (mod\ 7)$$

Which implies that $7 \mid N_k(F_{i+1})$.

Again if we take $n = 1$ then k will be 7, clearly we have-

$$N_{k=7}(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^{7} \equiv 0\ (mod\ 257)$$

which implies that $257 \mid N_k(F_{i+1})$.

In the same manner if we take $n = 2$ then k will be 11, clearly we have-

$$N_{k=11}(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^{11} \equiv 0\ (mod\ 11)$$

Which implies that $11 \mid N_k(F_{i+1})$. etc.

Above congruence shows that- If $k = 4n + 1$, then numbers of the form$N_k(F_{i+1})$may be divisible by some prime number.

6. **Theorem5: If $k = 3n$, where n is odd integer then numbers of the form$N_k(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^{k}$must be divisible by prime number 7.**

Proof: We have

$N_k(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^{k}$, $k \geq 1$.

Let $k = 3n,\ n = 1,2, \dots \dots \dots$ then-

$$N_{k=3n}(F_{i+1}) = \sum_{i=1}^{4} {F_{i+1}}^{3n}$$

clearly we have-

$\sum_{i=1}^{4} {F_{i+1}}^{3n} \equiv 0\ (mod\ 7)$.

This congruence implies that $7 \mid N_k(F_{i+1})$.

7. **Some numerical results**

Table- 1: We have displayed some numerical results of numbers $N_k(F_{i+1})$, for all $k = 1, 2, \dots \dots, 40$.

List of numbers $N_k(F_{i+1})$, $1 \leq k \leq 40$

Value of k	Value of $N_k(F_{i+1})$	Smallest Prime Factor	Nature of $N_k(F_{i+1})$
1	**11**	**1**	**Prime**
2	39	3	Composite
3	161	7	Composite
4	723	3	Composite
5	3,401	19	Composite
6	16,419	3	Composite
7	80,441	257	Composite

8	397,443	3	Composite
9	1,973,321	7	Composite
10	9,825,699	3	Composite
11	49,007,321	11	Composite
12	244,676,163	3	Composite
13	1,222,305,641	47	Composite
14	6,108,314,979	3	Composite
15	30,531959,801	7	Composite
16	152,631,002,883	3	Composite
17	**763,068,724,361**	**1**	**Prime**
18	3,815,084,948,259	3	Composite
19	19,074,649,113,881	41	Composite
20	95,370,919,473,603	3	Composite
21	476,847,620,653,481	7	Composite
22	2,384,217,176,269,539	3	Composite
23	11,921,023,106,645,561	19	Composite
24	59,604,927,221,704,323	3	Composite
25	298,024,071,199,117,001	23	Composite
26	1,490,118,661,317,702,819	3	Composite
27	7,450,588,222,655,530,841	7	Composite
28	37,252,925,861,680,031,043	3	Composite
29	186,264,583,554,009,938,921	53	Composite
30	931,322,780,507,684,352,099	3	Composite
31	4,656,613,490,752,936,345,721	11	Composite
32	2,328,306,621,841,144,670,9763	3	Composite
33	116,415,327,386,003,970,943,241	7	Composite
34	582,076,609,134,691,252,135,144	3	Composite
35	2,910,383,095,704,949,820,066,201	6489891697	Composite
36	14,551,915,378,461,555,823,116,483	3	Composite
37	72,759,576,592,118,302,363,153,961	1274850111973	Composite
38	363,797,882,060,023,287,716,914,659	3	Composite
39	1,818,989,407,598,412,178,604,868,281	7	Composite
40	9,094,947,029,886,948,937,718,947,203	3	Composite

8. Conclusion

In this research paper, theorems 1 & 2 shows that the numbers $N_k(F_{i+1})$ must be divisible by 3 when k is of the form $4n$ and $4n+2$. Theorem 5 shows that the numbers $N_k(F_{i+1})$ must be divisible by 7 when $k \equiv 3\ (mod\ 6)$ $i.e.$ $k = 6.n+3$, $n = 0,1, \dots\dots$.

Theorems 3 & 4 shows that the numbers $N_k(F_{i+1})$ may or may not be divisible by any prime when k is of the form $4n+1$ and $4n+3$. Moreover, we found out that the numbers of the form $N_k(F_{i+1})$ are composite in nature when the value of k is even, and mixed in nature when the value of k is odd. Finally we can say that the form $N_k(F_{i+1})$ are static nature when the value of k is even, and dynamic nature when the value of k is odd.

Acknowledgement

Authors are grateful to Professor. Hari Kishan, Department of Mathematics, D. N. College, Meerut for useful advice. Authors are also grateful to the reviewers for their valuable comments.

References

Atanasov, K. et al.: New Visual Perspectives on Fibonacci Numbers. World Scientific, Hackensack (2002).
David, M. Burton, Elementary Number Theory-McGraw Hill Publication-2010, 283- 30
Devlin, K., The man of numbers: Fibonacci's Arithmetic Revolution. New York, NY: Walker (2011).
Dunlap, RA: The Golden Ratio and Fibonacci Numbers. World Scientific, Hackensack (1997).
Garland, T. H., Fascinating Fibonaccis: Mystery and magic in numbers, Palo Alto, CA: Dale Seymour (1987).
Han, JS, Kim, HS, Neggers, J: The Fibonacci norm of a positive integer n- observations and conjectures. Int. J. Number Theory 6, (2010), 371-385.
Posamentier, A. S., & Lehmann, I., The Fabulous Fibonacci Numbers. Amherst, NY: Prometheus Books (2007).
Satish Kumar, Hari Kishan and Deepak Gupta, A Note on Multiplicative Triple Fibonacci Sequences, Bulletin of Society for Mathematical Services & Standards, Vol. 4 No. 1 (2014). 01- 06.
Satish Kumar, Hari Kishan and Deepak Gupta, Multiplicative Quadruple Fibonacci Sequences, International Transaction in Mathematical Science and Computers, Vol. 6 No. 2 (2014), July- December (2013), 183- 190.
T. Koshy, Fibonacci and Lucas's Numbers with applications, Wiley- Interscience, New York, NY, USA, (2001).