On regular graph,thathas two types of facesthat havingtwo consecutive numbers of edges

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Abstract:
It is known that any three - Regulargraph, planar and connected, and all the faces in it are pentagonal or hexagonal, thenthe number of pentagons in it is double the number of hexagonal edges, i.e. 12. [6].
The purpose of this paper is to prove the following:
Given ans-regular graph,planar and connected, and all the faces in it contain only (l)or (l+1)edges, also $l = \frac{s+2}{s-2}$, then if(k) marks the number of faces, which contain (l)edges, will be $2(l+1)$.
In fact, I will present here, (inan s-regular graph, planar and connected) thenecessary and sufficient general condition, whichcauses thatthe number of faces containing (l) edges to be doubleof the number (l+1), (I.e. equal to2(l+1)).
Keywords: Euler formula, regular graph, planar graphandconnected graph.

Introduction:
Wecalled a graph$d$-regular, if the degree of each vertex in the graphis$d$.
A basic law in graph theory guaranteesthatfor a graph$G{:}(V, E)$,the next is always true $\sum_{x \in V} \deg(x) = |E|$.
We define a connected graphto be the graph in which all two verticesin itbe connected in a path. Moreover, we also define;a planar graph is a graph thatwe can draw on the plane without cutting between its edges.
A face in a planar graph is a boundedareaof connected vertices with edges that unambiguously describe the face, that is, the boundary of a face.
Obviously, each planar graph has an outer face, which is usually marked withF_∞
We let $F(G)$denote the set of all faces of G, and $f = |F(G)|$.
Euler's formula states the following:
Given a planar and connected graph, and denote by n,m,$f$to the number of its vertexes, edges and facesrespectively, then it holds that, $n + f - m = 2$. [5,6].

New result:

The new result presented in the following theorem:

Theorem:

For all s-regular graph, planar and connected, and all the faces in it contain only (l)or (l+1) edges, and if (k) marks the number of faces, which contain (l)edges,then
$k = 2(l+1)$, if and only if $l = \frac{s+2}{s-2}$.

Proof:

Let $G = (V, E)$,an s-regular graph, planar and connected,and all the faces in it contain only (l)or (l+1) edges.

Denote by$n = |V|$and by $m = |E|$.let so,$f$denotethe number of all the faces of G.

We summarize the valuesdegreeof all the vertices in the graph$G$, in two different ways:

From the fact that thesum of the values degree of all the vertices in every graph, is always equal to twice the number of edges,we get that,

$\sum_{i=1}^{n} \deg(x_i) = 2|E| = 2m$,When $x_i \in V$ for all $1 \leq x_i \leq n$.

Furthermore, as it is given that the graph G, an s-regular graph, we get that,

$\sum_{i=1}^{n} \deg(x_i) = s \cdot n$.

From this, we will get:

$2m = \sum_{i=1}^{n} \deg(x_i) = s \cdot n$, so:

$$(I) \qquad n = \frac{2}{s}m$$

Denotenow in $F = \{F_1, F_2, F_3, \dots, F_f\}$ to the set ofall the f faces in G.denote also byt_{F_i} for the number of edges that participate in the construction of the face F_i, to all $F_i \in F$.

It is clear that$\sum_{F_i \in F} t_{F_i} = 2m$.

Now, if the graph has k faces each of thembounded by l edges, there will stay(f-k) faceseach of them boundedby l+1edges, hence

$\sum_{F_i \in F} t_{F_i} = k \cdot l + (f-k) \cdot (l+1)$,

Thus, we get that:

$2m = k \cdot l + (f-k) \cdot (l+1)$

Twice the number of edges,this is because each edge participates in exactly two faces.

If we isolate the f from the last equation, we get:

$$(II) \qquad f = \frac{2}{l+1}m + \frac{1}{l+1}k$$

We will set the value of n from$(a)$and the value of m from(b) in the Euler's formula:

$2 = n + f - m$.

We get:

$$2 = \frac{2}{s}m + \frac{2}{l+1}m + \frac{1}{l+1}k - m$$

$$(III) \qquad 2 = (\frac{2}{s} + \frac{2}{l+1} - 1)m + \frac{1}{l+1}k$$

To proceed with proof of the theorem, we must prove two directions:

First direction:

Let$l=\frac{s+2}{s-2}$, we will prove that $k=2(l+1)$.

If we set$l=\frac{s+2}{s-2}$in the factor of m in the last equation(III):

$$2=(\frac{2}{s}+\frac{2}{\frac{s+2}{s-2}+1}-1)m+\frac{1}{l+1}k$$

$$2=(\frac{2}{s}+\frac{2}{\frac{2s}{s-2}}-1)m+\frac{1}{l+1}k$$

$$2=(\frac{2}{s}+\frac{s-2}{s}-1)m+\frac{1}{l+1}k$$

$$2=(\frac{2+s-2-s}{s})m+\frac{1}{l+1}k$$

$$2=\frac{1}{l+1}k$$

$$2(l+1)=k$$

As required for proof of First direction.

Second direction:

Let $k=2(l+1)$, we will prove that$l=\frac{s+2}{s-2}$.

If we set $k=2(l+1)$ in the equation(III):

$$(III)\qquad 2=(\frac{2}{s}+\frac{2}{l+1}-1)m+\frac{1}{l+1}k$$

$$2=(\frac{2}{s}+\frac{2}{l+1}-1)m+\frac{1}{l+1}2(l+1)$$

$$2=(\frac{2}{s}+\frac{2}{l+1}-1)m+2$$

$$0=(\frac{2}{s}+\frac{2}{l+1}-1)m$$

$$0=\frac{2}{s}+\frac{2}{l+1}-1$$

$$0=\frac{2-s}{s}+\frac{2}{l+1}$$

$$\frac{s-2}{s}=\frac{2}{l+1}$$

$$(s-2)(l+1)=2s$$

$$l=\frac{s+2}{s-2}$$

As required for proof of Second direction.

This completes the proof of the theorem.

Corollary 1:

For s-regular graph, planar and connected, and all the faces in it contain only (l)or ($l+1$) edges, and if (k) marks the number of faces, which contain (l)edges, if$l = \frac{s+2}{s-2}$, then $k = \frac{4s}{s-2}$.

ProofCorollary 1:

If we set$l = \frac{s+2}{s-2}$ in equation(III):

$$(III) \qquad 2 = (\frac{2}{s} + \frac{2}{l+1} - 1)m + \frac{1}{l+1}k$$

$$2 = (\frac{2}{s} + \frac{2}{\frac{s+2}{s-2} + 1} - 1)m + \frac{1}{\frac{s+2}{s-2} + 1}k$$

$$2 = 0 \cdot m + \frac{1}{\frac{2s}{s-2}}k$$

$$2 = \frac{s-2}{2s}k$$

$$\frac{4s}{s-2} = k$$

This completes the proof of theCorollary 1.

Corollary 2:

There are only three different graphs in the terms of the previous theorem:

1. 3-regular graphand all the faces in it are pentagonal or hexagonal.
2. 4-regular graph and all the faces in it are Triangles and squares.
3. 6-regular graph and all the faces in it are Triangles and double edges.(It is clear that this graph is not simple, because it contains double edges).

ProofCorollary 2:

The following table describes the possibility for a graph$G$, s-regular, planar, connected, and all the faces in it contain only (l)or ($l+1$) edges, and $l = \frac{s+2}{s-2}$, and (k) marks the number of faces, which contain (l)edges.

s	$\boldsymbol{l = \frac{s+2}{s-2}}$	***l+1***	$\boldsymbol{k = 2(l+1)}$
1	abnormal		
2	abnormal		
3	5	6	12
4	3	4	8
5	abnormal		
6	2	3	6
7	abnormal		

8	abnormal		
⋮	⋮		

Since that, $lim_{s\to\infty}\left(\frac{s+2}{s-2}\right) = 1$, it is clear that there will be no other possiblevalues for*l*.
This completes the proof of the Corollary 2.

Remark:
To demonstratethe graph, which is, 6-regular and all the faces in it are triangles and double edges, look at the following two figures:

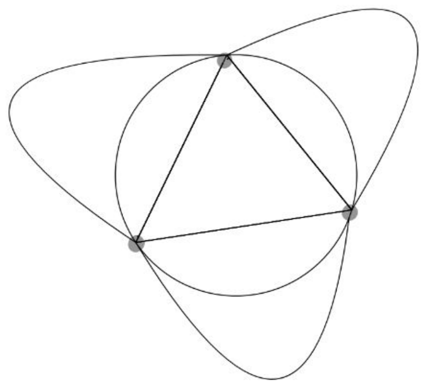

Figure 1

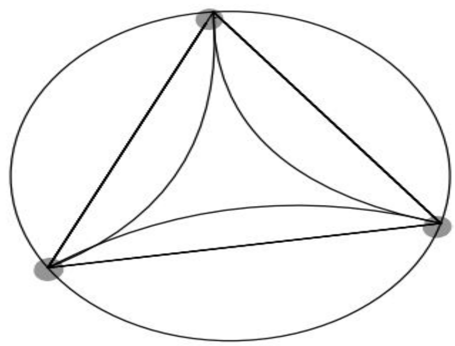

Figure 2

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