Perfect Dominating sets and Perfect Domination Polynomial of Some Standard Graphs

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ABSTRACT
The paper illustrates algebraic representation of the friendship Graph $F_n$and corona of $G$and K_1called the Perfect dominating polynomial. The Perfect dominating polynomial is constructed by using Perfect dominating set. At first we find the family Perfect dominating set with the given cardinality. The collection of families of sets become the coefficient of novel Perfect dominating polynomial . The relations which gets identified with this on coefficients helps to develop the Perfect dominating polynomial of $F_n$and $G \circ K_1$thus we find the rootsof this polynomial.

Keyword: Perfect Dominating set, Friendship Graph, Polynomial,Corona
2010 Mathematics Subject Classification*:* 05C31,05

1. Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n$. For any vertex $u \in V$, the open neighborhood of u is the set $N(u) = \{v \in V | uv \in E\}$. A set $S \subseteq V$ is a dominating set of G, if every vertex $u \in V$ is a element of$S$ or is adjacent to an element of S [7].The dominating set S is a perfect dominating set if $|N(u) \cap S| = 1$ for each $u \in V - S$[7], or equivalently, if every vertex u in $V - S$ is adjacent to exactly one vertex in S. The Perfect domination number γ_{pf} is the minimum cardinality of a Perfect dominating set in G.The Friendship Graph F_n is constructed by joining n copies of the cycle C_3 with a common vertex [4]. The corona $G_1 \circ G_2$ is obtained by taking one copy of G_1 and $|G_1|$ copies of G_2, and by joining each vertex of the ith copy of G_2 to the i^{th} vertex of G_i, $i = 1,2, \dots, |G_1|$[8].Let $D_{pf}(G, i)$ be the family of all Perfect dominating sets of G with cardinality i, and let $d_{pf}(G, i) = |D_{pf}(G, i)|$then
$D_{pf}(G, x) = \sum_{\gamma_{pf}(G)}^{|V(G)|} d_{pf}(G, i)\, x^i$is called the Perfect dominating polynomial of G.The roots of the polynomialis obtained byequate the given polynomial to zero and the roots are called the solutions for the given polynomial.

2.Perfect Dominating Polynomial of a Friendship Graph F_n

We denote the family of Perfect dominating sets of the Friendship Graph F_n with cardinality $i$by $D_{pf}(F_n, i)$. Then the Perfect dominating sets of the Friendship Graph F_n is investigated as follows;

Definition 2.1
Let F_n be a Friendship Graph with $2n + 1$ vertices and $D_{pf}(F_n, i)$be the family of Perfect dominating sets of the Friendship Graph F_n with cardinality i then, $d_{pf}(F_n, i) = |D_{pf}(F_n, i)|$.

Example 2.2
Consider the following Friendship Graph F_3 in Figure1

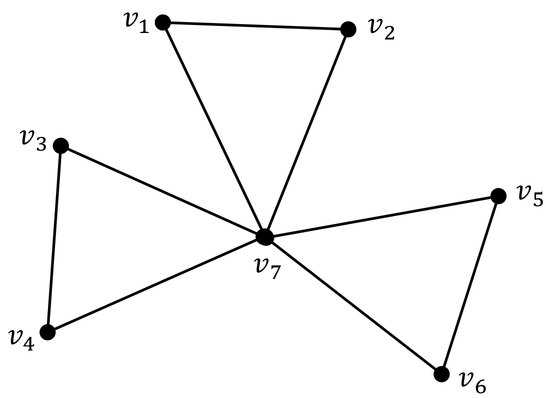

Figure:1

Here, the Perfect dominating set of cardinality one is $\{v_7\}$

The Perfect dominating set of cardinality two is { }
The Perfect dominating set of cardinality three is $\{\{v_1,v_2,v_7\},\{v_3,v_4,v_7\},\{v_5,v_6,v_7\}\}$
The Perfect dominating set of cardinality four is { }
The Perfect dominating set of cardinality five is $\{\{v_1,v_2,v_3,v_4,v_7\},\{v_1,v_2,v_5,v_6,v_7\},\{v_3,v_4,v_5,v_6,v_7\}\}$
The Perfect dominating set of cardinality six is { }
The Perfect dominating set of cardinality seven is $\{v_1,v_2,v_3,v_4,v_5,v_6,v_7\}$
Therefore, $d_{pf}(F_3,1)=1, d_{pf}(F_3,2)=0, d_{pf}(F_3,3)=3, d_{pf}(F_3,4)=0, d_{pf}(F_3,5)=3, d_{pf}(F_3,6)=0, d_{pf}(F_3,7)=1$.

Lemma 2.3

$d_{pf}(F_1,1)=3$

Proof

By the definition a friendship graph F_1 has 3 vertices and 3 edges.We know that, F_n is constructed by joining n copies of the cycle C_3 with a common vertex.Thus, we conclude that F_1 is a cycle with 3 vertices. Therefore every vertex inF_1 Perfectly dominates all the other two vertices in F_1 and we get $|D_{pf}(F_1,1))|=3$. Hence, $d_{pf}(F_1,1)=3$.

Theorem 2.4

Let F_n be a Friendship Graph with $2n+1$ vertices then,$d_{pf}(F_n,1)=1$ for $n\geq 2$

Proof

As the Graph $F_n$can be constructed by joining n copies of the cycle C_3 with a common vertex, which is the only vertex that Perfectly dominates all other vertex ofF_n $for\ n\geq 2$.Therefore, $d_{pf}(F_n,1)=1$ for $n\geq 2$

Lemma 2.5
$\gamma_{pf}(F_n)=1$
Lemma 2.6
Let F_n be the Friendship graph with $2n+1$ vertices and for all $n\geq 2$ $d_{pf}(F_n,i)=\begin{cases}\binom{n}{\left(\frac{i-1}{2}\right)} & for\ i=1,3,5,7,\dots,2n+1\\ 0 & otherwise\end{cases}$
Proof
Let F_n be the Friendship graph with $2n+1$ vertices and $3n$ edges.Since the Perfect dominating Set of the Friendship Graph $F_n$with cardinality $i$is obtained by choosing $\frac{i-1}{2}$ copies of the cycle C_3 that joins with a common vertex from n copies of the cycle C_3 joining with a common vertex, which is $\binom{n}{\left(\frac{i-1}{2}\right)}$Possible ways.Therefore, $d_{pf}(F_n,i)=\begin{cases}\binom{n}{\left(\frac{i-1}{2}\right)} & for\ i=1,3,5,7,\dots,2n+1\\ 0 & otherwise\end{cases}$
Definition 2.7
If F_n be a Friendship Graph with $2n+1$ vertices then $D_{pf}(F_n,x)=\sum_{i=1}^{2n+1}d_{pf}(F_n,i)x^i$ is called the Perfect Dominating Polynomial of F_n

Theorem 2.8
Let F_1 be a Friendship Graph with 3 vertices then the Perfect dominating Polynomial of F_1 is given by $D_{pf}(F_1,x)=3x+x^3$
Proof
Since, F_1 is a complete graph with 3 vertices then we have, $D_{pf}(F_1,x)=3x+x^3$
Theorem 2.9
Let F_n be a Friendship Graph with $2n+1$ vertices then the Perfect dominating Polynomial $D_{pf}(F_n,x)=x(1+x^2)^n$ for $n\geq 2$
Proof
Given F_n be a Friendship Graph with $2n+1$ vertices .We have $D_{pf}(F_n,x)=\sum_{i=1}^{2n+1}d_{pf}(F_n,i)x^i$

Then by lemma 2.6 we get $D_{pf}(F_n,x) = \binom{n}{0} x + \binom{n}{1} x^3 + \cdots + \binom{n}{n} x^{2n+1} = x\left(1 + \binom{n}{1} x^2 + \binom{n}{2} (x^2)^2 + \cdots + \binom{n}{n} (x^2)^n\right) = x(1 + x^2)^n$

Example 2.10
We find the Perfect dominating polynomial F_3, From Example 2.2 we have$d_{pf}(F_3,1) = 1, d_{pf}(F_3,2) = 0, d_{pf}(F_3,3) = 3, d_{pf}(F_3,4) = 0, d_{pf}(F_3,5) = 3, d_{pf}(F_3,6) = 0, d_{pf}(F_3,7) = 1$.
Then by definition 2.7$D_{pf}(F_3,x) = x + 3x^3 + 3x^5 + x^7$
By Theorem 2.9we have, $D_{pf}(F_3,x) = x(1 + x^2)^3 = x(1 + 3x^2 + 3x^4 + x^6) = x + 3x^3 + 3x^5 + x^7$
Theorem 2.11
The Perfect dominating roots of the friendship Graph F_n are given by 0and $\pm i$ $(n\ times)$.
Proof
The Perfect dominating Polynomial of a Friendship graph with n vertices is given by $D_{pf}(F_n,x) = x(1 + x^2)^n$
To find the roots of this polynomial put $D_{pf}(F_n,x) = 0$.
ie)$x(1 + x^2)^n = 0 \Rightarrow x = 0\ \ or\ (1 + x^2)^n = 0$
$\Rightarrow x = 0\ \ or\ = \pm i\ (n\ times)$

3. Perfect Dominating Polynomial of $G \circ K_1$

Definition 3.1

Let G be a Simple graphof order n and $D_{pf}(G \circ K_1, i)$ is a family of perfect dominating set with cardinality i and $d_{pf}(G \circ K_1, i) = |D_{pf}(G \circ K_1, i)|$then $D_{pf}(G \circ K_1, x) = \sum_{i=n}^{2n} d_{pf}(G \circ K_1, i) x^i$ is a perfect dominating polynomial of $G \circ K_1$.

Theorem 3.2

Let G be a Simple graph of order n then, $d_{pf}(G \circ K_1, i) = 0$ if $i < n$

Theorem 3.3

Let G be a Graph with n vertices then $\gamma_{pf}(G \circ K_1) = n$

Proof

Let $V(G) = \{u_1, u_2, \dots, u_n\}$ be the vertices of G and we add n new vertices $\{v_1, v_2, \dots, v_n\}$ to G and join v_i to u_i for all $i, 1 \le i \le n$ to obtain $G \circ K_1$.Let D be a Perfect dominating set of G then $|D| \le n$ here we have two cases if $|D| < n$ then $|N(v_i) \cap D| \ne 1\ for\ some\ i, 1 \le i \le n$, therefore D is not a perfect dominating set of $G \circ K_1$.If $|D| = n$ then $D = V(G)$ and $|N(v_i) \cap D| = 1\ for\ every\ i, 1 \le i \le n$. therefore, $V(G) = \{u_1, u_2, \dots, u_n\}$ is a Perfect dominating set of $G \circ K_1$.Hence, $\gamma_{pf}(G \circ K_1) = n$

Theorem 3.4

Let G be a graph of order n then, $d_{pf}(G \circ K_1, n) = 2$

Proof

We take $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V'(G) = \{v_1, v_2, \dots, v_n\}$ be the set of n vertices and join v_i to u_i for all $i, 1 \le i \le n$ to obtain $G \circ K_1$.Let D be a family of perfect dominating set cardinality n of $G \circ K_1$First we claim there is no perfect dominating sets belongs to D with the combination of vertices $V(G)$ and $V'(G)$. If not Suppose $V_1 \in D$ and vertices of V_1 belongs to $V(G)$ and $V'(G)$ then, $|N(u_i) \cap D| \ne 1\ for\ some\ i, 1 \le i \le n$.Which contradicts to the definition of perfect dominating Set. Hence, there is no perfect dominating sets belongs to D with the combination of vertices $V(G)$ and $V'(G)$.But, for the set $V(G)$ which is a dominating set also $|N(v_i) \cap V(G)| = 1\ for\ all\ i, 1 \le i \le n$ and $V'(G)$ is also a dominating set and $|N(u_i) \cap V'(G)| = 1\ for\ all\ i, 1 \le i \le n$.Therefore, $V(G), V'(G) \in D$.Hence, $d_{pf}(G \circ K_1, n) = |D| = 2$.

Theorem 3.5

Let G be a graph of order n and for every m where $n < m \le 2n$ we have $d_{pf}(G \circ K_1, m) = \binom{n}{m-n}$.

Proof

Let G be a graph of order n and D is a Perfect dominating Set of $G \circ K_1$ with size m where $n < m \le 2n$ then $|D \cap V(G)| = i$ for $1 \le i \le n$.With out loss of generality Suppose that $V(G) \cap D = \{u_1, u_2, \dots, u_n\}$. Since, D is a

Perfect dominating set with size m then, D contains some $v_i, v_{i+1} \dots, v_n$ vertices. Hence, for finding the dominating set D we have to extend $\{u_1, u_2, \dots, u_n\}$ to $\{u_1, u_2, \dots, u_n, v_i, v_{i+1} \dots, v_n\}$.Which is of $\binom{n}{i}$ possibilities therefore $d_{pf}(G \circ K_1, m) = \binom{n}{m-n}$for $n < m \leq 2n$.

Theorem 3.6

Let G be a graph of order n then $D_{pf}(G \circ K_1, x) = x^n[1 + (1 + x^n)^n]$

Proof

We have $D_{pf}(G \circ K_1, x) = \sum_{i=n}^{2n} d_{pf}(G \circ K_1, i)\, x^i$ that is$D_{pf}(G \circ K_1, x) = d_{pf}(G \circ K_1, n)x^n + d_{pf}(G \circ K_1, n+1)x^{n+1} + \cdots + d_{pf}(G \circ K_1, 2n)x^{2n}$. By using Theorem 3.4& Theorem 3.5we get $D_{pf}(G \circ K_1, x) = 2x^n + \binom{n}{1}x^{n+1} + \binom{n}{2}x^{n+2} + \cdots + \binom{n}{n-1}x^{n+(n-1)} + \binom{n}{n}x^{2n}$.$= 2x^n + x^n(1+x)^n - x^n$.$= x^n[1 + (1+x)^n]$

Example 3.7

Consider a graph G of order 4 then the corona of two graphs G and K_1 is $G \circ K_1$ has 8 vertices

Hence, by a Theorem 3.6 we have $D_{pf}(G \circ K_1, x) = x^4[1 + (1 + x)^4]$

$$= x^4[1 + (1 + 4x + 6x^2 + 4x^3 + x^4)]$$

$$= x^4[2 + 4x + 6x^2 + 4x^3 + x^4]$$

$$= 2x^4 + 4x^5 + 6x^6 + 4x^7 + x^8$$

Theorem 3.8

The Perfect dominating roots of $G \circ K_1$ are $0$and $\left[cos\frac{(2k+1)\pi}{n} + isin\frac{(2k+1)\pi}{n}\right] - 1, k = 0,1,2, \dots, n-1$.

Proof

The Perfect dominating Polynomial of a $G \circ K_1$ with $2n$ vertices is given by $D_{pf}(G \circ K_1, x) = x^n[1 + (1 + x)^n]$.To find the roots of this polynomial put $D_{pf}(G \circ K_1, x) = 0$ therefore, $x^n[1 + (1 + x)^n] = 0 \Rightarrow x^n = 0$ or $1 + (1 + x)^n = 0$.Now, $(1 + x)^n = -1 \Rightarrow (1 + x)^n = 1(cos\pi + isin\pi) \Rightarrow 1 + x = 1^{\frac{1}{n}}(cos\pi + isin\pi)^{\frac{1}{n}} \Rightarrow x = \left[cos\frac{(2k+1)\pi}{n} + isin\frac{(2k+1)\pi}{n}\right] - 1, k = 0,1,2, \dots, n-1$. Therefore, the Perfect dominating roots of $G \circ K_1$ are $0(n\ times)$and $\left[cos\frac{(2k+1)\pi}{n} + isin\frac{(2k+1)\pi}{n}\right] - 1, k = 0,1,2, \dots, n-1$.

Example 3.9

Let G be a graph of order 2 we have to find the Perfect dominating roots of $G \circ K_1$.By the previous theorem if G is a graph of order n then $0(n\ times)$ and $\left[cos\frac{(2k+1)\pi}{n} + isin\frac{(2k+1)\pi}{n}\right] - 1, k = 0,1,2, \dots, n-1$ are the Perfect dominating roots of the polynomial. Put $n = 2$ we get $0(2\ times)$ and $\left[cos\frac{(2k+1)\pi}{2} + isin\frac{(2k+1)\pi}{2}\right] - 1, k = 0,1$ that is$0(2\ times)$ and $\left[cos\frac{\pi}{2} + isin\frac{\pi}{2}\right] - 1, \left[cos\frac{3\pi}{2} + isin\frac{3\pi}{2}\right] - 1$,therefore $0(2\ times)$ and -1 , $-i - 1$ are the required Perfect dominating roots of the Graph.

4. Conclusions

The paper sums up the findings of how perfect dominating polynomial of a Friendship Graph and $G \circ K_1$ is structured up by Perfect dominating set and also how this polynomialsobtain its roots.

Acknowledgement

I express my deep sense of gratitude to my Lord, the Almighty for his Mercy and Grace, which I have received bountifully. I owe my deep sense of gratitude and profound thankfulness to my respectable Research guide and Colleagues.

References

[1] A.M Anto, P.Paul Hawkins and T Shyla Isac Mary, Perfect Dominating Sets and Perfect Dominating Polynomial of a Cycle, Advances in Mathematics: Scientific Journal, 8(3)(2019), 538–543.

[2] A.M Anto, P.Paul Hawkins and T Shyla Isac Mary, Perfect Dominating Sets and Perfect Dominating Polynomial of a Path, International Journal of Advanced Science and Technology, 28(16)(2010), 1226-1236

[3] Bondy J.A, Murthy. U.S.R., *Graph theory with applications*, Elsevier Science Publishing Co, sixth printing, 1984
[4] Gallian, J. A, "Dynamic Survey DS6: Graph Labeling" *Electronic Journal of Combinatorics*, DS6, 1-58
[5] Gray Chartand, Ping Zhang, 2005, *Introduction to graph theory*, McGraw Hill, Higher Education
[6] S. Alikhani and Y.H. Peng, *Introduction to Domination polynomial of a graph*, arXiv: 0905.225 [v] [math.co] 14 may (2009)
[7] T.W.Haynes, S.T.Hedetniemi, and P.J.Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Newyork(1998)
[8] Y. N. Yeh and I. Gutman, On the sum of all distances in composite graphs, Discrete Math., 135 (1994) 359–365