An Inequality for a Polynomial and its Derivative

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Abstract: If $P(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n not vanishing in $|z| < K, K \leq 1$, then Govil_[Proc. Natl. Acad. Sci., 50(1980), pp.50-52] proved

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+K^n}\max_{|z|=1}|P(z)|$$

provided $|P'(z)|$ and $|Q'(z)|$ attain their maximum at the same point on $|z| = 1$, where $Q(z) = z^n\overline{P(1/\bar{z})}$. In this paper an improvement of above inequality is obtained and an extension to the s^{th} derivative for polynomials of degree $n \geq 2$ is proved.

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1. Introduction

Let *P(z)* be a polynomial of degree *n* and *P'(z)* its derivative. According to a well known result due to S. Bernstein [1], we have

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)|. \tag{1.1}$$

The result is best possible and equality holds for $(z) = \alpha z^n$, where $|\alpha| = 1$.

If we restrict to the class of polynomials having no zero in $|z| < 1$, then the bound in inequality (1.1) can be improved. In this direction, P. Erdös [2] conjectured and later Lax [6] verified that if $p(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1.2}$$

The above result is best possible and equality holds for $P(z) = \alpha+\beta z^n$, where $|\alpha| = |\beta|$.

While seeking for generalization of inequality (1.2), Malik [7] considered the polynomial $p(z) \neq 0$ in $|z| < K, K \geq 1$, and proved

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+K}\max_{|z|=1}|P(z)|. \tag{1.3}$$

The above result is best possible and equality holds for the polynomial $P(z) = (z+K)^n$.

For polynomials not vanishing in $|z| < K, K \leq 1$, Govil [4] proved the following result analogous to (1.3).

Theorem A Let $P(z) = \sum_{j=1}^{n} c_j z^j$ is a polynomial of degree n, having no zero in $|z| < K$, $K \leq 1$ and $Q(z) = z^n \overline{P(1/\bar{z})}$. If $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z| = 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+K^n} \max_{|z|=1} |P(z)|. \tag{1.4}$$

2. Main Results

In this paper, we first obtain an improvement of the bound in Theorem A by involving some coefficients of $P(z)$.

Theorem 1. Let $P(z) = \sum_{j=1}^{n} c_j z^j$ be a polynomial of degree $n \geq 2$, having no zero in $|z| < K$, $K \leq 1$ and let $Q(z) = z^n \overline{P(1/\bar{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain the maximum at the same point on $|z| = 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+K^n} \max_{|z|=1} |P(z)| - \left\{\frac{1-K^4}{1+K^n}\right\} |c_{n-1}| \quad for \;\; n > 2, \tag{2.5}$$

and

$$\max_{|z|=1} |P'(z)| \leq \frac{2}{1+K^2} \max_{|z|=1} |P(z)| - \left\{\frac{1-K^2}{1+K^2}\right\} |c_1| \quad for \;\; n = 2. \tag{2.6}$$

It is clearly of interest to obtain a generalization of Theorem 1 for the s^{th} derivative of $P(z)$. In this direction, we prove

Theorem 2. Let $P(z) = \sum_{j=1}^{n} c_j z^j$ be a polynomial of degree $n \geq 2$, having no zero in $|z| < K$, $K \leq 1$ and let $Q(z) = z^n \overline{P(1/\bar{z})}$. If $|P^{(s)}(z)|$ and $|Q^{(s)}(z)|$ attain the maximum at the same point on $|z| = 1$, then

$$\max_{|z=1|} |P^{(s)}(z)| \leq \frac{n(n-1)\ldots(n-s+1)}{1+K^n} \max_{|z=1|} |P(z)| - \left\{\frac{1-K^4}{1+K^n}\right\} |c_{n-s}| \quad for \;\; n-s > 2, \tag{2.7}$$

and

$$\max_{|z=1|}|P^{(s)}(z)| \leq \frac{n(n-1)\dots(n-s+1)}{1+K^n}\max_{|z=1|}|P(z)| - \left\{\frac{1-K^2}{1+K^2}\right\}|c_{n-s}| \quad for\ \ n-s=1. \tag{2.8}$$

Remark 1. Theorem 2 reduces to Theorem 1 if we take *s=1* in Theorem 2.

3. Lemmas

For the proof of the theorems, we need the following lemmas.

Lemma 1. If *P(z)* is a polynomial of degree n, then for $|z|=1$

$$|P^{(s)}(z)| + |Q^{(s)}(z)| \leq n(n-1)\dots(n-s+1)\max_{|z|=1}|P(z)|\,, \tag{3.1}$$

where $Q(z) = z^n\overline{P(1/\bar{z})}$.

The above lemma is due to Govil and Rahman [5]. The next lemma is due to Frappier et al.[3].

Lemma 2. If $P(z) = \sum_{j=1}^{n} c_j z^j$ is a polynomial of degree n, then for *R>1*

$$\max_{|z|=R}|P(z)| \leq R^n \max_{|z=1|}|P(z)| - (R^n - R^{n-2})|P(0)| \quad for\ \ n \geq 2\,, \tag{3.2}$$

and

$$\max_{|z|=R}|P(z)| \leq R\max_{|z|=1}|P(z)| - (R-1)|P(0)| \quad for\ \ n=1\,. \tag{3.3}$$

The coefficient of $|P(0)|$ is best possible for each *R*.

Lemma 3. If $P(z) = \sum_{j=1}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in the disk $|z| < K, K \leq 1$, then for $1 \leq s < n$

$$\max_{|z|=1}|Q^{(s)}(z)| \leq K^n \max_{|z|=1}|P^{(s)}(z)| - (K^n - K^{n-4})|c_s| \quad for\ \ n-s>2 \tag{3.4}$$

and

$$\max_{|z|=1}|Q^{(s)}(z)| \leq K^n \max_{|z|=1}|P^{(s)}(z)| - (K^n - K^{n-2})|c_s| \quad for\ \ n-s=1, \tag{3.5}$$

where $(z) = z^n\overline{P(1/\bar{z})}$.

Proof of lemma 3. Since all the zeros of the polynomial *P(z)* lie in $|z| < K, K \leq 1$, therefore the polynomial $G(z) = P(Kz)$ has all its zeros in the disk $|z| < 1$. Let $H(z) = z^n\overline{G(1/\bar{z})}$, then $H(z) = z^n\overline{P(K/\bar{z})} = K^nQ(z/K)$ has all its zeros in $|z| > 1$. Therefore, $G(z)/H(z)$ is analytic in $|z| < 1$ and $|G(z)| = |H(z)|$ for $|z| = 1$. It follows by maximum modulus principle that $|G(z)| \le |H(z)|$ for $|z| < 1$. Replacing z by $1/z$, we get $|H(z)| \le |G(z)|$ for $|z| \ge 1$. For every real or complex number λ with $|\lambda| > 1$, we get $|H(z)| \le |\lambda G(z)|$ for $|z| \ge 1$. It follows by Rouche's theorem that the polynomial $H(z) - \lambda G(z)$ has all its zeros in $|z| < 1$. Hence by Gauss-Lucas theorem, the polynomial

$$H^{(s)}(z) - \lambda G^{(s)}(z) \tag{3.6}$$

has all its $(n-s)$ zeros in $|z| < 1$, which implies

$$\left|H^{(s)}(z)\right| \le \left|G^{(s)}(z)\right| \quad for \quad |z| \ge 1. \tag{3.7}$$

If inequality (3.7) is not true , then there must be a point $z = z_0$ with $|z_0| \ge 1$ such that

$$\left|H^{(s)}(z_0)\right| > \left|G^{(s)}(z_0)\right|.$$

We take

$$\lambda = \frac{H^{(s)}(z_0)}{G^{(s)}(z_0)}$$

so that $|\lambda| > 1$ and from (3.6) , with this choice of λ, we get $H^{(s)}(z_0) - \lambda G^{(s)}(z_0) = 0$ for $|z_0| \ge 1$, which contradicts the fact that all the zeros of the polynomial $H^{(s)}(z) - \lambda G^{(s)}(z)$ lie in $|z| < 1$. Hence for $|z| \ge 1$, the inequality (3.7) is true. Substituting $G(z) = P(Kz)$ and $H(z) = K^nQ(z/K)$ in inequality (3.7), we get

$$K^{n-s}\left|Q^{(s)}(z/K)\right| \le K^s\left|P^{(s)}(Kz)\right| \qquad |z| \ge 1. \tag{3.8}$$

Putting Kz in place of z in inequality (3.8), we get

$$K^{n-2s}\left|Q^{(s)}(z)\right| \le \left|P^{(s)}(K^2z)\right| \qquad |z| \ge 1. \tag{3.9}$$

In particular, from inequality (3.9), we have

$$K^{n-2s}\max_{|z|=1}\left|Q^{(s)}(z)\right| \le \max_{|z|=K^2}\left|P^{(s)}(K^2z)\right|. \tag{3.10}$$

For the case $n - s \ge 2$.

Using inequality (3.2) of Lemma 2 to the polynomial $P^{(s)}(z)$ of degree $n - s \ge 2$ with $= K^2 \ge 1$, we obtain

$$\max_{|z|=K^2}\left|P^{(s)}(z)\right| \le (K^2)^{n-s}\max_{|z|=1}\left|P^{(s)}(K^2z)\right| - ((K^2)^{n-s} - (K^2)^{n-s-2})\left|P^{(s)}(0)\right| . \tag{3.11}$$

Combing the inequalities (3.10) and (3.11), we get

$$K^{n-2s}\max_{|z|=1}\left|Q^{(s)}\right| \le K^{2n-2s}\max_{|z|=1}\left|P^{(s)}\right| - (K^{2n-2s} - K^{2n-2s-4})|c_s|,$$

from which follows inequality (3.4).

For the case $n - s = 1$.
Inequality (3.10), becomes

$$K^{2-n} \max_{|z|=1} \left|Q^{(s)}(z)\right| \leq \max_{|z|=K^2} \left|P^{(s)}(K^2 z)\right|. \tag{3.12}$$

Using inequality (3.3) of Lemma 2 to the polynomial $P^{(s)}(z)$ of degree $\boldsymbol{n - s = 1}$ for $R = K^2 \geq 1$, we get

$$\max_{|z|=K^2} \left|P^{(s)}(z)\right| \leq K^2 \max_{|z|=1} \left|P^{(s)}(K^2 z)\right| - (K^2 - 1)\left|P^{(s)}(0)\right|. \tag{3.13}$$

On combining inequality (3.12) and (3.13), we get

$$K^{n-2s} \max_{|z|=1} \left|Q^{(s)}\right| \leq K^2 \max_{|z|=1} \left|P^{(s)}\right| - (K^2 - 1)\left|P^{(s)}(0)\right|,$$

from which inequality (3.5) follows.

Lemma 4 If $P(z) = \sum_{j=1}^{n} c_j z^j$ is a polynomial of degree n having no zero in the disk $|z| < K, K \leq 1$, then for $1 \leq \boldsymbol{s} < n$,

$$K^n \max_{|z|=1} \left|P^{(s)}(z)\right| \leq \max_{|z|=1} Q^{(s)}(z) - (1 - K^4)|c_{n-s}| \quad for \quad n - s \geq 2, \tag{3.14}$$

and

$$K^n \max_{|z|=1} \left|P^{(s)}(z)\right| \leq \max_{|z|=1} Q^{(s)}(z) - (1 - K^2)|c_{n-s}| \quad for \quad n - s = 1, \tag{3.15}$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$.

Proof of lemma 4 Since $P(z) = \sum_{j=1}^{n} c_j z^j$ has no zero in $|z| < K, K \leq 1$, then the polynomial $Q(z) = z^n \overline{P(1/\bar{z})}$ has all its zeros in $|z| \leq \frac{1}{K}, \frac{1}{K} \geq 1$.
For the case $n - s \geq 2$.
Applying (3.4) of Lemma 3 to the polynomial *Q(z)*, we get

$$\max_{|z|=1} \left|P^{(s)}(z)\right| \leq \left(\frac{1}{K}\right)^n \max_{|z|=1} \left|Q^{(s)}(z)\right| - \left\{\left(\frac{1}{K}\right)^n - \left(\frac{1}{K}\right)^{n-4}\right\} |c_{n-s}|,$$

from which inequality (3.14) follows.
For the case $n - s = 1$. Applying inequality (3.5) of Lemma 3 to the polynomial *Q(z)*, we get

$$\max_{|z|=1} \left|P^{(s)}(z)\right| \leq \left(\frac{1}{K}\right)^n \max_{|z|=1} \left|Q^{(s)}(z)\right| - \left\{\left(\frac{1}{K}\right)^n - \left(\frac{1}{K}\right)^{n-2}\right\} |c_{n-s}|,$$

from which follows the inequality (3.15).

4. Proof of the theorems

As Theorem 2 is a generalization of Theorem 1, therefore here we give the proof of Theorem 2 only.

Proof of Theorem 2. Since by hypothesis, $|P^{(s)}(z)|$ and $|Q^{(s)}(z)|$ attain the maximum at the same point on unit circle, we choose a point z_0 on the unit circle such that $|P^{(s)}(z_0)| = \max_{|z|=1}|P^{(s)}(z)|$ and $|Q^{(s)}(z_0)| = \max_{|z|=1}|Q^{(s)}(z)|$, then by inequality (3.1) of Lemma 1, we have

$$|P^{(s)}(z_0)| + |Q^{(s)}(z_0)| \leq n(n-1)\dots(n-s+1)\max_{|z|=1}|P(z)|\,,$$

i.e.

$$\max_{|z|=1}|P^{(s)}(z)| + \max_{|z|=1}|Q^{(s)}(z)| \leq n(n-1)\dots(n-s+1)\max_{|z|=1}|P(z)|\,. \qquad (4.1)$$

Case *n-s>2*.
On combining inequality (3.14) of Lemma 4 with (4.1), we get

$$\max_{|z|=1}|P^{(s)}(z)| + K^n\max_{|z|=1}|P^{(s)}(z)| + (1-K^4)|c_{n-s}| \leq n(n-1)\dots(n-s+1)\max_{|z|=1}|P(z)|\,,$$

which is equivalent to

$$\max_{|z|=1}|P^{(s)}(z)| \leq \frac{n(n-1)\dots(n-s+1)}{1+K^n}\max_{|z|=1}|P(z)| - \left\{\frac{1-K^4}{1+K^n}\right\}|c_{n-s}|$$

and the inequality (2.7) follows.

Case *n-s=1*. In this case, combining inequality (3.15) of Lemma 4 with inequality (4.1), we get

$$\max_{|z|=1}|P^{(s)}(z)| + K^n\max_{|z|=1}|P^{(s)}(z)| + (1-K^2)|c_{n-s}| \leq n(n-1)\dots(n-s+1)\max_{|z|=1}|P(z)|\,,$$

which gives

$$\max_{|z|=1}|P^{(s)}(z)| \leq \frac{n(n-1)\dots(n-s+1)}{1+K^n}\max_{|z|=1}|P(z)| - \left\{\frac{1-K^2}{1+K^n}\right\}|c_{n-s}|$$

and this follows the inequality (2.8).

References

[1] Bernstein S., "Lecons sur_ les propriétés extrémales et la meilleure approximation desfonctions analytiques d'une variable réelle ", Gauthier Villars Paris, 1926.

[2] Erdös P., "On extremal properties of the derivative of polynomials", Ann. of Math., (2) 41 (1940), pp. 310-313.

[3] Frappier C. , Rahman_Q.R. and St. Ruscheweyh , "New Inequalities for Polynomials", Transaction of the American Mathematical Society, Vol. 288 (1985), pp. 69-99.

[4] Govil N.K., "On a theorem of S. Bernstein", Proc. Natl. Acad. Sci., 50 (1980), pp. 50-52.

[5] Govil N.K. and Rahman_Q.I., "Functions of exponential type not vanishing in a half plane and related polynomails " Trans. Amer. Math. Soc., 137 (1969), pp.501-517.

[6] Lax P.D., "Proof of a conjecture of P. Erdös on the derivatives of a polynomial", Bull. Amer. Math. Soc., 50 (1944), pp. 509-513.

[7] Malik M.A., "On the derivative of a polynomial", J. London Math. Soc., 2 (1) (1969), pp.57-60.